6 Applications of Derivatives

Fastrack Revision

▶ **Rate of Change**

- $\frac{dy}{dx}$ = rate of change of y w.r.t. x.
- $\left[\frac{dy}{dx}\right]_{x=x_0}$ denotes the rate of change of y w.r.t. x at $x = x_0$.
- When x and y are varying w.r.t. t then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ if $\frac{dx}{dt} \neq 0$, so $\frac{dy}{dx}$ = rate of change of y w.r.t. x $= \frac{\text{rate of change of } y \text{ w.r.t. } t}{\text{rate of change of } x \text{ w.r.t. } t}$
- If $C(x)$ represents the cost function for x units produced, then marginal cost is given by
$$MC = \frac{d}{dx}\{C(x)\}$$
- If $R(x)$ represents the revenue function for x units sold, then marginal revenue is given by
$$MR = \frac{d}{dx}\{R(x)\}$$

▶ **Decreasing and Increasing Functions**

- A real valued function $f(x)$, is called:
  - (i) increasing in open interval (a, b), if for all
  $$x_1, x_2 \in (a, b): x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$
  - (ii) strictly increasing in open interval (a, b), if for all $x_1, x_2 \in (a, b): x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
  - (iii) decreasing in open interval (a, b) if for all
  $$x_1, x_2 \in (a, b): x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$
  - (iv) strictly decreasing in open interval (a, b) if for all $x_1, x_2 \in (a, b): x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
  - (v) monotonically if $f(x)$ is either increasing or decreasing in the interval (a, b).
- A real valued function $f(x)$ is called increasing or decreasing in closed interval $[a, b]$ if $f(x)$ is increasing or decreasing in open interval (a, b) and increasing or decreasing at $x = a$ and $x = b$.
- Let a real differentiable function $f(x)$ is defined in open interval (a, b).
  - (i) If for each $x \in (a, b)$, $f'(x) \geq 0$, then $f(x)$ is increasing in the interval (a, b).
  - (ii) If for each $x \in (a, b)$, $f'(x) > 0$, then $f(x)$ is strictly increasing in the interval (a, b).
  - (iii) If for each $x \in (a, b) f'(x) \leq 0$, then $f(x)$ is decreasing in the interval (a, b).
  - (iv) If for each $x \in (a, b)$, $f'(x) < 0$, then $f(x)$ is strictly decreasing in the interval (a, b).

▶ **Maxima and Minima**

- **Absolute Maximum Value of a Function:** A function $f(x)$ is said to have the greatest value or absolute maximum value at point 'c' in its domain, if $f(x) \leq f(c)$ for all x in the domain of $f(x)$. In this case, 'c' is called the point of maxima.
- **Absolute Minimum Value of a Function:** A function $f(x)$ is said to have the smallest value or absolute minimum value at a point 'c' in its domain, if $f(x) \geq f(c)$ for all x in the domain of $f(x)$. In this case, 'c' is called the point of minima.
- **Local Maximum Value of a Function:** Let 'c' is a point of local maximum of a function $f(x)$ if there exists an $h > 0$ such that $f(x) < f(c)$ for all $c - h, c + h$. In this case, $f(c)$ is called a local maximum value of $f(x)$.
- **Local Minimum Value of a Function:** Let 'c' is a point of local minimum of a function $f(x)$ if there exists an $h > 0$ such that $f(x) > f(c)$ for all $c - h, c + h$. In this case, $f(c)$ is called a local minimum value of $f(x)$.
- **Stationary Points (Turning Points or Critical Points):** The points at which $f'(c) = 0$ are known as stationary points or turning points or critical points.
- **Inflection Point:** If $f'(x)$ does not change sign as x increases through c, then x is neither a point of maximum nor a point of minimum, then $x = c$ is called a point of inflection.
- **First Derivative Test:** Let $f(x)$ be a differentiable function on an interval I and $c \in I$. Then,
  - (i) Point c is a local maximum of $f(x)$, if
    - (a) $f'(c) = 0$
    - (b) $f'(x) > 0$, if $x \in (c - h, c)$ and $f'(x) < 0$, if $x \in (c, c + h)$, where h is a small but positive quantity.
  - (ii) Point c is a local minimum of $f(x)$, if
    - (a) $f'(c) = 0$
    - (b) $f'(x) < 0$, if $x \in (c - h, c)$ and $f'(x) > 0$, if $x \in (c, c + h)$, where h is a small but positive quantity.
  - (iii) If $f'(c) = 0$ but $f'(x)$ does not changes sign in $(c - h, c + h)$ for any positive quantity, then $x = c$ is neither a point of minimum nor a point of maximum.
- **Second Derivative Test:** Let f be a function defined on a interval I and $c \in I$. Let f be twice differentiable at c. Then,
  - (i) $x = c$ is a point of local maximum if $f'(c) = 0$ and $f''(c) < 0$. The value of $f(c)$ is local maximum value of f.
  - (ii) $x = c$ is a point of local minimum if $f'(c) = 0$ and $f''(c) > 0$. The value of $f(c)$ is local minimum value of f.
  - (iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to the first derivative test and find whether c is a point of maxima, minima or a point of inflection.

Practice Exercise

Multiple Choice Questions

Q 1. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 15 cm, then the rate at which the thickness of ice decreases, is:

a. $\frac{5}{6\pi}$ cm/min b. $\frac{1}{54\pi}$ cm/min
c. $\frac{1}{18\pi}$ cm/min d. $\frac{1}{36\pi}$ cm/min

Q 2. The rate of change of the surface area of a sphere of radius r, when the radius is increasing at the rate of 2 cm/s is proportional to:

a. $\frac{1}{r}$ b. $\frac{1}{r^2}$ c. r d. r^2

Q 3. The sides of an equilateral triangle are increasing at the rate of 2 cm/s. The rate at which the area increases, when the side is 20 cm, is:

a. $\sqrt{3}$ cm^2/s b. 20 cm^2/s
c. $20\sqrt{3}$ cm^2/s d. $\frac{20}{\sqrt{3}}$ cm^2/s

Q 4. If the volume of a sphere is increasing at a constant rate, then the rate at which its radius is increasing, is:

a. a constant
b. proportional to the radius
c. inversely proportional to the radius
d. inversely proportional to the surface area

Q 5. A ladder 10 m long rests against a vertical wall with the lower end on the horizontal ground. The lower end of the ladder is pulled along the ground away from the wall at the rate of 3 cm/s. The height of the upper end while it is descending at the rate of 4 cm/s, is:

a. $4\sqrt{3}$ m b. $5\sqrt{3}$ m c. 6 m d. 8 m

Q 6. The value of b for which the function $f(x) = x + \cos x + b$ is strictly decreasing over R, is: *(CBSE SQP 2021 Term-1)*

a. $b < 1$ b. No value of b exists
c. $b \le 1$ d. $b \ge 1$

Q 7. The real function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is: *(CBSE SQP 2021 Term-1)*

a. strictly increasing in $(-\infty, -2)$ and strictly decreasing in $(-2, \infty)$
b. strictly decreasing in $(-2, 3)$
c. strictly decreasing in $(-\infty, 3)$ and strictly increasing in $(3, \infty)$
d. strictly decreasing in $(-\infty, -2) \cup (3, \infty)$

Q 8. Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is strictly increasing: *(CBSE SQP 2021 Term-1)*

a. $(-\infty, 2) \cup (2, \infty)$ b. $(2, \infty)$
c. $(-\infty, 2)$ d. $(-\infty, 2) \cup (2, \infty)$

Q 9. The function $f(x) = x^3 + 3x$ is increasing in the interval: *(CBSE 2023)*

a. $(-\infty, 0)$ b. $(0, \infty)$ c. R d. $(0, 1)$

Q 10. The function $f(x) = 2x^3 - 15x^2 + 36x + 6$ is increasing in the interval: *(CBSE SQP 2021 Term-1)*

a. $(-\infty, 2) \cup (3, \infty)$ b. $(-\infty, 2)$
c. $(-\infty, 2] \cup [3, \infty)$ d. $[3, \infty)$

Q 11. The function $y = x^2 e^{-x}$ is decreasing in the interval: *(CBSE SQP 2021 Term-1)*

a. $(0, 2)$ b. $(2, \infty)$
c. $(-\infty, 0)$ d. $(-\infty, 0) \cup (2, \infty)$

Q 12. $f(x) = \left(\frac{e^{2x} - 1}{e^{2x} + 1}\right)$ is:

a. an increasing function b. a decreasing function
c. an even function d. None of these

Q 13. If $f(x) = x^3 - 6x^2 + 9x + 3$ is a decreasing function, then x lies in:

a. $(-\infty, -1) \cap (3, \infty)$ b. $[1, 3]$
c. $(3, \infty)$ d. None of these

Q 14. The interval in which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing, is: *(CBSE 2023)*

a. $(-1, \infty)$ b. $(-2, -1)$
c. $(-\infty, -2)$ d. $[-1, 1]$

Q 15. Function $f(x) = \frac{\lambda \sin x + 6\cos x}{2\sin x + 3\cos x}$ is increasing, if:

a. $\lambda > 1$ b. $\lambda < 1$ c. $\lambda < 4$ d. $\lambda > 4$

Q 16. The function $f(x) = x^3 + 6x^2 + (9 + 2k)x + 1$ is strictly increasing for all x, if:

a. $k > \frac{3}{2}$ b. $k \ge \frac{3}{2}$ c. $k < \frac{3}{2}$ d. $k \le \frac{3}{2}$

Q 17. The function $f(x) = x^2 - 2x$ is strictly decreasing in the interval:

a. $(-\infty, 1)$ b. $[1, \infty)$
c. R d. None of these

Q 18. The function $f(x) = \log(1 + x) - \frac{2x}{2 + x}$ is increasing on:

a. $(-1, \infty)$ b. $(-\infty, 0)$
c. $(-\infty, \infty)$ d. None of these

Q 19. For what value of a, $f(x) = -x^3 + 4ax^2 + 2x - 5$ is decreasing for all x?

a. ± 5 b. 3
c. 0 d. No value of a

Q 20. The function $f(x) = 2\log(x - 2) - x^2 + 4x + 1$ increases on the interval:

a. (1, 2) b. (2, 3) c. (1, 3) d. (2, 4)

Q 21. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing?

a. $(0, 1)$ b. $\left(\frac{\pi}{2}, \pi\right)$
c. $\left(0, \frac{\pi}{2}\right)$ d. None of these

Q 22. The maximum value of $[x(x-1)+1]^{1/3}, 0 \le x \le 1$ is: *(CBSE SQP 2021 Term-1)*

a. 0 b. $\frac{1}{2}$ c. 1 d. $\sqrt[3]{\frac{1}{3}}$

Q 23. The area of a trapezium is defined by function f and given by $f(x)=(10+x)\sqrt{100-x^2}$, then the area when it is maximised is: *(CBSE SQP 2021 Term-1)*

a. 75 cm^2 b. $7\sqrt{3}$ cm^2
c. $75\sqrt{3}$ cm^2 d. 5 cm^2

Q 24. The maximum value of $\left(\frac{1}{x}\right)^x$ is: *(CBSE SQP 2021 Term-1)*

a. $e^{1/e}$ b. e c. $\left(\frac{1}{e}\right)^{1/e}$ d. e^e

Q 25. The least value of the function $f(x)=2\cos x+x$ in the closed interval $\left[0,\frac{\pi}{2}\right]$ is: *(CBSE SQP 2021 Term-1)*

a. 2
b. $\frac{\pi}{6}+\sqrt{3}$
c. $\frac{\pi}{2}$
d. the least value does not exist

Q 26. In a sphere of radius r, a right circular cone of height h having maximum curved surface area is inscribed. The expression for the square of curved surface of cone is: *(CBSE SQP 2021 Term-1)*

a. $2\pi^2 rh(2rh+h^2)$ b. $\pi^2 hr(2rh+h^2)$
c. $2\pi^2 r(2rh^2-h^3)$ d. $2\pi^2 r^2(2rh-h^2)$

Q 27. The absolute maximum value of the function $f(x)=4x-\frac{1}{2}x^2$ in the interval $\left[-2,\frac{9}{2}\right]$ is:

a. 8 b. 9 c. 6 d. 10

Q 28. A function $f: R \to R$ is defined as $f(x)=x^3+1$. Then, the function has: *(CBSE SQP 2021 Term-1)*

a. no minimum value
b. no maximum value
c. Both maximum and minimum values
d. Neither maximum value nor minimum value

Q 29. It is given that at $x=1$, the function x^4-62x^2+ax+9 attains its maximum value on the interval [0, 2]. Find the value of a.

a. 100 b. 120
c. 140 d. 160

Q 30. The function $f(x)=x+\frac{4}{x}$ has:

a. local maxima at $x=2$ and local minima at $x=-2$
b. local minima at $x=2$ and local maxima at $x=-2$
c. absolute maxima at $x=2$ and absolute minima at $x=-2$
d. absolute minima at $x=2$ and absolute maxima at $x=-2$

Q 31. Amongst all pairs of positive numbers with product 256, find those whose sum is the least.

a. 16, 14 b. 16, 16 c. 64, 4 d. 32, 8

Q 32. $\sin^p\theta\cos^q\theta$ attains a maximum, when $\theta=$

a. $\tan^{-1}\sqrt{\frac{p}{q}}$ b. $\tan^{-1}\left(\frac{p}{q}\right)$
c. $\tan^{-1}q$ d. $\tan^{-1}\left(\frac{q}{p}\right)$

Q 33. If $f(x)=\frac{1}{4x^2+2x+1}$, then its maximum value is:

a. 3/4 b. 8/3
c. 4/3 d. 3/8

Q 34. The maximum value of the function $y=x(x-1)^2$ is:

a. 0 b. $\frac{4}{27}$
c. -4 d. None of these

Q 35. The minimum value of $\frac{x}{\log x}$ is:

a. e b. $1/e$
c. 1 d. None of these

Q 36. If $f(x)=2x^3-21x^2+36x-30$, then which one of the following is correct?

a. $f(x)$ has minimum at $x=1$
b. $f(x)$ has maximum at $x=6$
c. $f(x)$ has maximum at $x=1$
d. $f(x)$ has no maximum or minimum

Q 37. A missile is fired from the ground level rises x metre vertically upwards in t second, where $x=100t-\frac{25}{2}t^2$. The maximum height reached is:

a. 200 m b. 125 m c. 160 m d. 190 m

Q 38. If the function $f(x)=x^2e^{-2x}$, $x>0$, then the maximum value of $f(x)$ is:

a. $\frac{1}{e}$ b. $\frac{1}{2e}$ c. $\frac{1}{e^2}$ d. $\frac{4}{e^4}$

Q 39. If the two positive numbers x and y such that $x+y=60$ and x^3y is maximum, then:

a. $x=15, y=45$ b. $x=45, y=15$
c. $x=30, y=30$ d. $x=20, y=40$

Q 40. Let $f(x)=\cos\pi x+10x+3x^2+x^3$, $x\in[-2,3]$. Then, the absolute minimum value of $f(x)$ is:

a. 0 b. -15
c. $3-2\pi$ d. None of these

Q 41. Two towns A and B are 60 km apart. A school is to be built to serve 150 students in town A and 50 students in town B. If the total distance to be travelled by 200 students is to be as small as possible, then the school be built at:

a. town B b. 45 km from town A
c. town A d. 45 km from town B

Q 42. The value of a, so that the sum of the squares of the roots of the equation $x^2 - (a-2)x - a + 1 = 0$ assume the least value, is:

a. 2 b. 0 c. 3 d. 1

Assertion & Reason Type Questions

Directions (Q. Nos. 43-53): *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)
b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)
c. Assertion (A) is true but Reason (R) is false
d. Assertion (A) is false but Reason (R) is true

Q 43. Assertion (A): Both $\sin x$ and $\cos x$ are decreasing functions in $\left(\frac{\pi}{2}, \pi\right)$.

Reason (R): If a differentiable function decreases in (a, b), then its derivative also decreases in (a, b).

Q 44. Assertion (A): The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$.

Reason (R): The function x^2e^x and x^2e^{-x} are increasing for all $x > 0$ and the sum of two increasing functions in any interval (a, b) is an increasing function in (a, b).

Q 45. Assertion (A): If the function $f(x) = \frac{ae^x + be^{-x}}{ce^x + de^{-x}}$ is increasing function of x, then $bc > ad$.

Reason (R): A function $f(x)$ is increasing if $f'(x) > 0$ for all x.

Q 46. Let $f: R \to R$ be differentiable and strictly increasing function throughout its domain.

Assertion (A): If $|f(x)|$ is also strictly increasing function, then $f(x) = 0$ has no real roots.

Reason (R): At ∞ or $-\infty$, $f(x)$ may approach to 0, but cannot be equal to zero.

Q 47. Assertion (A): Let $f: R \to R$ be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$. Then, f is an increasing function.

Reason (R): If $f'(x) < 0$, then $f(x)$ is a decreasing function.

Q 48. Assertion (A): $f(x) = xe^{-x}$ has maximum at $x = 1$.

Reason (R): $f'(1) = 0$ and $f''(1) < 0$.

Q 49. Assertion (A): The graph $y = x^3 + ax^2 + bx + c$ has extremum, if $a^2 < 3b$.

Reason (R): A function, $y = f(x)$ has an extremum, if $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$ for all $x \in R$.

Q 50. Let $f(x)$ be a polynomial function of degree 6 such that $\frac{d}{dx}(f(x)) = (x-1)^3 (x-3)^2$, then

Assertion (A): $f(x)$ has a minimum at $x = 1$.

Reason (R): When $\frac{d}{dx}(f(x)) < 0, \ \forall\, x \in (a-h, a)$ and $\frac{d}{dx}(f(x)) > 0, \forall\, x \in (a, a+h)$; where '$h$' is an infinitesimally small positive quantity, then $f(x)$ has a minimum at $x = a$, provided $f(x)$ is continuous at $x = a$. *(CBSE SQP 2023-24)*

Q 51. Assertion (A): Let $f(x) = 5 - 4(x-2)^{2/3}$, then at $x = 2$, the function $f(x)$ attains neither least value nor greatest value.

Reason (R): $x = 2$ is the only critical point of $f(x)$.

Q 52. Assertion (A): The absolute maximum and minimum values of $f(x) = x^2\sqrt{1+x}$ in $\left[-1, \frac{1}{2}\right]$ are $\frac{\sqrt{6}}{8}$ and 0 respectively.

Reason (R): Let f be a differentiable function on I and x_0 be any interior point of I. If f attains its absolute maximum or minimum value at x_0, then $f'(x_0) = 0$.

Q 53. Assertion (A): The absolute maximum value of the function $2x^3 - 24x$ in the interval [1, 3] is 89.

Reason (R): The absolute maximum value of the function can be obtained from the value of the function at critical points and at boundary points.

Answers

1. (c)	2. (c)	3. (c)	4. (d)	5. (c)	6. (b)	7. (b)	8. (b)	9. (c)	10. (c)
11. (d)	12. (a)	13. (b)	14. (b)	15. (d)	16. (a)	17. (a)	18. (a)	19. (d)	20. (b)
21. (d)	22. (c)	23. (c)	24. (a)	25. (c)	26. (c)	27. (a)	28. (d)	29. (b)	30. (b)
31. (b)	32. (a)	33. (c)	34. (b)	35. (a)	36. (c)	37. (a)	38. (c)	39. (b)	40. (b)
41. (c)	42. (d)	43. (c)	44. (c)	45. (d)	46. (a)	47. (b)	48. (a)	49. (a)	50. (a)
51. (c)	52. (a)	53. (d)							

Case Study Based Questions

Case Study 1

The fuel cost per hour for running a train is proportional to the square of the speed it generates in km per hour. If the fuel cost ₹ 48 per hour at speed 16 km/h and the fixed charges to run the train amount to ₹ 1200 per hour.

Assume the speed of the train as v km/h.

Based on the above information, solve the following questions. **(CBSE SQP 2021 Term-1)**

Q 1. Given that the fuel cost per hour is k times the square of the speed the train generates in km/h, the value of k is:

a. $\frac{16}{3}$ b. $\frac{1}{3}$ c. 3 d. $\frac{3}{16}$

Q 2. If the train has travelled a distance of 500 km, then the total cost of running the train is given by function:

a. $\frac{15}{16}v + \frac{600000}{v}$ b. $\frac{375}{4}v + \frac{600000}{v}$

c. $\frac{5}{16}v^2 + \frac{150000}{v}$ d. $\frac{3}{16}v + \frac{6000}{v}$

Q 3. The most economical speed to run the train is:

a. 18 km/h b. 5 km/h
c. 80 km/h d. 40 km/h

Q 4. The fuel cost for the train to travel 500 km at the most economical speed is:

a. ₹ 3750 b. ₹ 750
c. ₹ 7500 d. ₹ 75000

Q 5. The total cost of the train to travel 500 km at the most economical speed is:

a. ₹ 3750 b. ₹ 75000
c. ₹ 7500 d. ₹ 15000

Solutions

1. Given, fuel cost = ₹ 48/hour

and speed = 16 km/hour.

According to question, the fuel cost per hour is k times the square of the speed the train generates in km/h.

i.e., Fuel cost $= k(\text{speed})^2$

$$\Rightarrow \quad 48 = k(16)^2$$

$$\Rightarrow \quad k = \frac{48}{16 \times 16} = \frac{3}{16}$$

So, option (d) is correct.

2. Total cost of running train (let C)

$$= \text{Fuel cost} + \text{Fixed charge} = \frac{3}{16}v^2t + 1200t$$

Distance covered = 500 km

$$\Rightarrow \quad \text{time} = \frac{500}{v} \text{ hrs}$$

Total cost of running train 500 km

$$= \frac{3}{16}v^2\left(\frac{500}{v}\right) + 1200\left(\frac{500}{v}\right)$$

$$\Rightarrow \quad C = \frac{375}{4}v + \frac{600000}{v}$$

So, option (b) is correct.

3. We have, $C = \frac{375}{4}v + \frac{600000}{v}$

Differentiate w.r.t. v, we get

$$\frac{dC}{dv} = \frac{375}{4} - \frac{600000}{v^2}$$

For economical speed of train,

$$\frac{dC}{dv} = \frac{375}{4} - \frac{600000}{v^2} = 0$$

$$\Rightarrow \quad v^2 = \frac{600000 \times 4}{375}$$

$$\Rightarrow \quad v^2 = 6400 \text{ km/h}$$

$$\Rightarrow \quad v = 80 \text{ km/h}$$

So, option (c) is correct.

4. Fuel cost for the train to travel 500 km

$$= \frac{375}{4}v$$

$\therefore$ Fuel cost for most economical speed $= \frac{375}{4} \times 80$

$= ₹\,7500$

So, option (c) is correct.

5. Total cost of the train to travel 500 km

$$= \frac{375}{4}v + \frac{600000}{v}$$

$\therefore$ Total cost for most economical speed

$$= \frac{375}{4} \times 80 + \frac{600000}{80}$$

$= 7500 + 7500 = ₹\,15000$

So, option (d) is correct.

Case Study 2

In a residential society comprising of 100 houses, there were 60 children between the ages of 10-15 years. They were inspired by their teachers to start composting to ensure that biodegradable waste is recycled. For this purpose, instead of each child doing it for only his/her house, children convinced the Residents welfare association to do it as a society initiative. For this, they identified a square area in the local park. Local authorities charged amount of ₹ 50 per square metre for space so that there is no misuse of the space and Resident welfare association takes it seriously. Association hired a labourer for digging out 250 m^3 and he charged ₹ 400 × (depth)^2. Association will like to have minimum cost.

Based on the given information, solve the following questions. ***(CBSE SQP 2021 Term-1)***

Q 1. Let side of square plot is x metre and its depth is h metre, then cost c for the pit is:

a. $\frac{50}{h} + 400h^2$ b. $\frac{12500}{h} + 400h^2$

c. $\frac{250}{h} + h^2$ d. $\frac{250}{h} + 400h^2$

Q 2. Value of h (in m) for which $\frac{dc}{dh} = 0$ is:

a. 1.5 b. 2 c. 2.5 d. 3

Q 3. $\frac{d^2c}{dh^2}$ is given by:

a. $\frac{25000}{h^3} + 800$ b. $\frac{500}{h^3} + 800$

c. $\frac{100}{h^3} + 800$ d. $\frac{500}{h^3} + 2$

Q 4. Value of x (in m) for minimum cost is:

a. 5 b. $10\sqrt{\frac{5}{3}}$ c. $5\sqrt{5}$ d. 10

Q 5. Total minimum cost of digging the pit (in ₹) is:

a. 4100 b. 7500 c. 7850 d. 3220

Solutions

1. Let each side of square shaped pit be x metre and depth as h metre.

As volume of earth taken out = 250 m^3

$\Rightarrow \quad x \cdot x \cdot h = 250 \text{ m}^3$

$\Rightarrow \quad x^2 = \frac{250}{h}$ ...(1)

Since, local authorities charged amount of ₹ 50 per square metre for space.

$\therefore$ Cost of land = ₹ 50 × space for land

= ₹ 50 × x^2

= ₹ 50 × $\frac{250}{h}$ = ₹ $\frac{12500}{h}$ [from eq. (1)]

So, total cost = cost of land + cost of labour

$\Rightarrow \quad c = \frac{12500}{h} + 400 \times (\text{depth})^2$

$\Rightarrow \quad c = \frac{12500}{h} + 400h^2$ ...(2)

So option (b) is correct.

2. Now, $\frac{dc}{dh} = \frac{d}{dh}\left\{\frac{12500}{h} + 400h^2\right\}$

$= -\frac{12500}{h^2} + 800h$ ...(3)

$\because \quad \frac{dc}{dh} = 0$ [given]

$\Rightarrow \quad \frac{-12500}{h^2} + 800h = 0$

$\Rightarrow \quad 800h^3 = 12500$ [$\because h \neq 0$]

$\Rightarrow \quad h^3 = \frac{125}{8} = \left(\frac{5}{2}\right)^3$

$\Rightarrow \quad h = 2.5\text{ m}$

So, option (c) is correct.

3. Differentiate eq. (3) w.r.t. h, we get

$\frac{d^2c}{dh^2} = \frac{d}{dh}\left\{-\frac{12500}{h^2} + 800h\right\}$

$= \frac{25000}{h^3} + 800$

So, option (a) is correct.

4. Now, value of x (in m) for minimum cost

$= \sqrt{\frac{250}{h}}$ [from eq. (1)]

$\left[\because \left.\frac{d^2c}{dh^2}\right|_{at\, h=2.5} > 0\right]$

$= \sqrt{\frac{250}{2.5}}$ [$\because h = 2.5$]

$= \sqrt{100} = 10$

So, option (d) is correct.

5. Total minimum cost of digging the pit (in ₹)

$= \frac{12500}{h} + 400h^2$ [from eq. (2)]

$= \frac{12500}{2.5} + 400(2.5)^2$ [$\because h = 2.5$]

= 5000 + 2500 = ₹ 7500

So, option (b) is correct.

Case Study 3

Let f be continuous on $[a, b]$ and differentiable on the open interval (a, b) then:

(a) f is strictly increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$.

(b) f is strictly decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$.

(c) f is a constant function in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

Based on the above information, solve the following questions:

Q 1. The function $f(x) = \cos(x)$ is strictly increasing in:

a. $(\pi, 2\pi)$ b. $(0, \pi)$ c. $\left(\frac{\pi}{2}, \pi\right)$ d. $(0, 2\pi)$

Q 2. The function $f(x) = 3x + 17$ is strictly increasing in:

a. R_- b. R_0 c. R d. Z_0

Q 3. The function $f(x) = \sin(x)$ is:

a. strictly increasing in $\left(0, \frac{\pi}{2}\right)$

b. strictly decreasing in $\left(0, \frac{\pi}{2}\right)$

c. strictly increasing in $\left(\frac{\pi}{2}, \pi\right)$

d. None of the above

Q 4. The function $f(x) = e^{2x}$ is strictly increasing on:

a. only $Z_{\oplus}$ b. only $R_{\oplus}$

c. R d. only R_{-}

Q 5. The function $f(x) = \log(\sin(x))$ is strictly increasing on:

a. $\left(0, \frac{\pi}{2}\right)$ b. $\left(\frac{\pi}{2}, \pi\right)$

c. $\left(-\frac{\pi}{2}, 0\right)$ d. None of these

Solutions

1. Given, $f(x) = \cos x$

Then, $f'(x) = -\sin x$

In interval $(\pi, 2\pi)$,

$f'(x) > 0$ $[\because \sin x < 0]$

Therefore, $f(x)$ is strictly increasing on $(\pi, 2\pi)$.

So, option (a) is correct.

2. Given, $f(x) = 3x + 17$

Differentiate w.r.t. x, we get

$f'(x) = 3 > 0$, in every interval of R.

Thus, the function is strictly increasing on R.

So, option (c) is correct.

3. Given function is $f(x) = \sin x$.

Differentiate w.r.t. x, we get

$f'(x) = \cos x$

Since, for each $x \in \left(0, \frac{\pi}{2}\right)$, $\cos x > 0$; therefore we have

$f'(x) > 0$ $[\because \cos x$ is positive in first quadrant$]$

Hence, f is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

So, option (a) is correct.

4. Given, $f(x) = e^{2x}$

Differentiate w.r.t. x, we get

$f'(x) = 2e^{2x} > 0$ in every interval of R.

Thus, the function is strictly increasing on R.

So, option (c) is correct.

5. Given, $f(x) = \log \sin x$

Then, $f'(x) = \frac{1}{\sin x} \times \cos x = \cot x$

In interval $\left(0, \frac{\pi}{2}\right)$,

$f'(x) > 0$

Therefore, $f(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

So, option (a) is correct.

Case Study 4

An architecture design an auditorium for a school for its cultural activities. The floor of the auditorium is rectangular in shape and has a fixed perimeter P.

Based on the above information, solve the following questions:

Q 1. If x and y represents the length and breadth of the rectangular region, then relation between the variable is:

a. $x + y = P$ b. $x^2 + y^2 = P^2$

c. $2(x + y) = P$ d. $x + 2y = P$

Q 2. The area (A) of the rectangular region, as a function of x, can be expressed as:

a. $A = Px + \frac{x}{2}$ b. $A = \frac{Px + x^2}{2}$

c. $A = \frac{Px - 2x^2}{2}$ d. $A = \frac{x^2}{2} + Px^2$

Q 3. School's manager is interested in maximising the area of floor 'A' for this to be happen, the value of x should be:

a. P b. $\frac{P}{2}$ c. $\frac{P}{3}$ d. $\frac{P}{4}$

Q 4. The value of y, for which the area of floor is maximum, is:

a. $\frac{P}{2}$ b. $\frac{P}{3}$ c. $\frac{P}{4}$ d. $\frac{P}{16}$

Q 5. Maximum area of floor is:

a. $\frac{P^2}{16}$ b. $\frac{P^2}{64}$ c. $\frac{P^2}{4}$ d. $\frac{P^2}{28}$

Solutions

1. Perimeter of floor = 2 (length + breadth)

$\Rightarrow$ $P = 2(x + y)$

So, option (c) is correct.

2. Area, A = length $\times$ breadth

$\Rightarrow$ $A = xy$ ...(1)

Since, $P = 2(x + y)$

$\Rightarrow$ $\frac{P - 2x}{2} = y$

From eq. (1), $A = xy$

$\Rightarrow$ $A = x\left(\frac{P - 2x}{2}\right)$

$\Rightarrow$ $A = \frac{Px - 2x^2}{2}$

So, option (c) is correct.

3. We have, $A = \frac{1}{2}(Px - 2x^2)$

$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{2}(P - 4x) = 0$

For maximum or minimum of A,

$\Rightarrow \quad \frac{dA}{dx} = 0$

$\Rightarrow \quad P - 4x = 0 \Rightarrow x = \frac{P}{4}$

Clearly, at $x = \frac{P}{4}, \frac{d^2A}{dx^2} = -2 < 0$

$\therefore$ Area is maximum at $x = \frac{P}{4}$.

So, option (d) is correct.

4. We have, $y = \frac{P - 2x}{2} = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$ $\quad \left[\because x = \frac{P}{4}\right]$

So, option (c) is correct.

5. We have, $A = xy = \frac{P}{4} \cdot \frac{P}{4} = \frac{P^2}{16}$ (maximum area)

So, option (a) is correct.

Case Study 5

The relation between the height of the plant ('y' in cm) with respect to its exposure to the sunlight is governed by the following equation $y = 4x - \frac{1}{2}x^2$, where 'x' is the number of days exposed to the sunlight, for $x \leq 3$.

Based on the above information, solve the following questions: *(CBSE SQP 2023-24)*

Q 1. Find the rate of growth of the plant with respect to the number of days exposed to the sunlight.

Q 2. Does the rate of growth of the plant increase or decrease in the first three days? What will be the height of the plant after 2 days?

Solutions

1. Given equation is $y = 4x - \frac{1}{2}x^2$.

The rate of growth of the plant with respect to the number of days exposed is

$$\frac{dy}{dx} = \left(4 - \frac{2x}{2}\right) = 4 - x$$

2. $\because \frac{dy}{dx} = 4 - x$

At $x = 1, 2, 3$; $\frac{dy}{dx} > 0$

Hence, the rate of growth of the plant increase in first three days.

$\because \quad y = 4x - \frac{1}{2}x^2$

At $x = 2$,

$$y = 4 \times 2 - \frac{1}{2}(2)^2$$
$$= 8 - 2 = 6$$

Case Study 6

The shape of a toy is given as $f(x) = 8x^4 - 4x^2 + 3$.

To make the toy beautiful 2 sticks which are perpendicular to each other were placed at a point (4, 5) above the toy.

Based on the above information, solve the following questions:

Q 1. Find the abscissa of the critical point of the function $f(x)$.

Q 2. Find the maximum value of the function.

Q 3. At which of the following intervals will $f(x)$ be decreasing?

Solutions

1. For the critical point of $f(x)$, $f'(x) = 0$

$\Rightarrow \quad \frac{d}{dx}(8x^4 - 4x^2 + 3) = 0$

$\Rightarrow \quad 32x^3 - 8x = 0$

$\Rightarrow \quad 8x(4x^2 - 1) = 0$

$\Rightarrow \quad x = 0$ or $4x^2 - 1 = 0$

$\Rightarrow \quad x = 0$ or $x^2 = \frac{1}{4}$

$\Rightarrow \quad x = 0$ or $x = \pm\frac{1}{2}$

Thus, abscissa of critical points are 0 and $\pm\frac{1}{2}$.

2. $\because \quad f'(x) = 32x^3 - 8x$

$\therefore \quad f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}(32x^3 - 8x) = 96x^2 - 8$

At $x = \pm\frac{1}{2}$,

$$f''\left(\pm\frac{1}{2}\right) = 96\left(\pm\frac{1}{2}\right)^2 - 8$$
$$= 96 \times \frac{1}{4} - 8 = 24 - 8 = 16 > 0$$

At $x = 0$,

$f''(0) = 96(0)^2 - 8 = 0 - 8 = -8 < 0$

So, the function is maximum at $x = 0$ and minimum at $x = \pm\frac{1}{2}$.

$\therefore$ Maximum value of function is

$$f(0) = 8(0)^4 - 4(0)^2 + 3 = 0 - 0 + 3 = 3$$

3. ∵ $f(x) = 8x^4 - 4x^2 + 3$

∴ $f'(x) = 32x^3 - 8x$

Putting, $f'(x) = 0$, we get,

$$32x^3 - 8x = 0$$

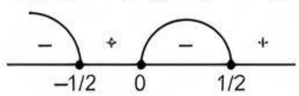
$\Rightarrow$ $8x(4x^2 - 1) = 0$

$\Rightarrow$ $8x(2x - 1)(2x + 1) = 0$

$\Rightarrow$ $x = 0, -\frac{1}{2}$ and $\frac{1}{2}$

which divides real line into four intervals

$$\left(-\infty, \frac{-1}{2}\right), \left(\frac{-1}{2}, 0\right), \left(0, \frac{1}{2}\right) \text{ and } \left(\frac{1}{2}, \infty\right).$$



Therefore, $f(x)$ is decreasing in $\left(-\infty, \frac{-1}{2}\right)$ and $\left(0, \frac{1}{2}\right)$

i.e., $\left(-\infty, \frac{-1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

Case Study 7

$P(x) = -5x^2 + 125x + 37500$ is the total profit function of a company, where x is the production of the company.

Based on the above information, solve the following questions:

Q 1. What will be the production when the profit is maximum?

Q 2. What will be the maximum profit?

Or

When the production is 2 units, what will be the profit of the company?

Q 3. Find the interval in which the profit function is strictly increasing.

Solutions

1. Given, $P(x) = -5x^2 + 125x + 37,500$

$P'(x) = -10x + 125$

$P''(x) = -10$

For maximum or minimum, we have

$P'(x) = 0$

$\Rightarrow$ $-10x + 125 = 0$

$\Rightarrow$ $x = 12.5$

Clearly, $P''(x) < 0$ for $x = 12.5$

Thus, profit is maximum when $x = 12.5$.

2. Maximum profit is given by,

$P(12.5) = -5(12.5)^2 + 125(12.5) + 37,500$

$= -781.25 + 1562.5 + 37,500$

$=$ ₹ 38,281.25

Or

$P(2) = -5(2)^2 + 125(2) + 37500$

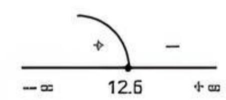
$= -20 + 250 + 37500 =$ ₹ 37730

3. Given, profit function $P(x) = -5x^2 + 125x + 37500$

Differentiate w.r.t. x, we get

$P'(x) = -10x + 125x$

$= -10(x - 12.5)$



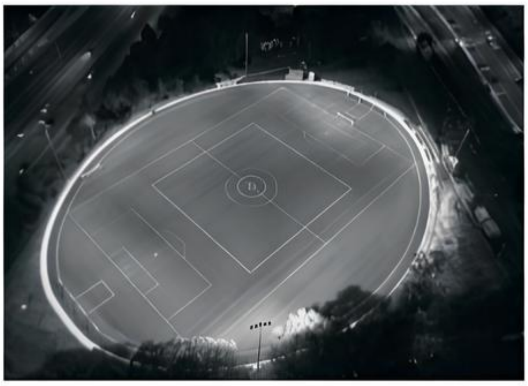
Since, for each $x \in (-\infty, 12.5)$

$P'(x) > 0$

Hence, $P(x)$ i.e., profit function is strictly increasing in $(-\infty, 12.5)$.

Case Study 8

In an elliptical sport field, the authority wants to design a rectangular soccer field with the maximum possible area. The sport field is given by the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Based on the above information, solve the following questions: **(CBSE SQP 2022-23)**

Q 1. If the length and the breadth of the rectangular field are $2x$ and $2y$ respectively, then find the area function in terms of x.

Q 2. Find the critical point of the function.

Q 3. Use first derivative test to find the length $2x$ and width $2y$ of the soccer field (in terms of a and b) that maximise its area.

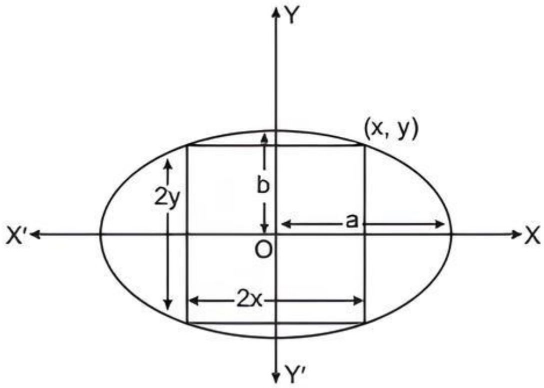
Or

Use Second Derivative Test to find the length $2x$ and width $2y$ of the soccer field (in terms of a and b) that maximise its area.

Solutions

1. Given equation of elliptical sport field is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(1)}$$



Here, a = length of semi-major axis

and b = length of semi-minor axis

From eq. (1),

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\Rightarrow \quad y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad \text{[for first quadrant]}$$

Let $(x, y) = \left(x, \frac{b}{a}\sqrt{a^2 - x^2}\right)$ be the upper right vertex of the rectangle.

$\therefore$ The area function A = length × breadth

$$= 2x \times 2y = 4x\frac{b}{a}\sqrt{a^2 - x^2}$$

$$= \frac{4b}{a}x\sqrt{a^2 - x^2}, x \in (0, a) \qquad ...(2)$$

2. Now differentiating eq. (2) w.r.t. x, we get

$$\frac{dA}{dx} = \frac{4b}{a}\left[\sqrt{a^2 - x^2} + x \times \frac{-2x}{\sqrt{a^2 - x^2}} \times \frac{1}{2}\right]$$

$$= \frac{4b}{a}\left\{\frac{a^2 - x^2 - x^2}{\sqrt{a^2 - x^2}}\right\}$$

$$= \frac{-4b}{a} \times \frac{2\left(x^2 - \frac{a^2}{2}\right)}{\sqrt{a^2 - x^2}} = \frac{-8b}{a} \times \frac{\left(x + \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right)}{\sqrt{a^2 - x^2}}$$

For maximum or minimum of A,

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \quad -\frac{8b}{a} \times \frac{\left(x + \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right)}{\sqrt{a^2 - x^2}} = 0$$

$$\Rightarrow \quad x = \frac{a}{\sqrt{2}} \qquad \left[\because x \neq \pm a, -\frac{a}{\sqrt{2}} \text{ i.e., can't be negative}\right]$$

So, $x = \frac{a}{\sqrt{2}}$ is the critical point.

3. For the values of x less than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}$,

$$\frac{dA}{dx} > 0$$

and for the values of x greater than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}$,

$$\frac{dA}{dx} < 0$$

Hence, by using the first derivative test, there is a local maximum at the critical point $x = \frac{a}{\sqrt{2}}$. Since, there is only one critical point, therefore the area of the soccer field is maximum at this critical point $x = \frac{a}{\sqrt{2}}$.

Thus, for maximum area of the soccer field,

length of the soccer field $= 2x = 2 \times \frac{a}{\sqrt{2}} = a\sqrt{2}$

and width of the soccer field $= 2y = \frac{2b}{a}\sqrt{a^2 - x^2}$

$$= \frac{2b}{a}\sqrt{a^2 - \frac{a^2}{2}} = \frac{2b}{a} \times \frac{a}{\sqrt{2}} = b\sqrt{2}.$$

Or

From part (1), $A = \frac{4b}{a}x\sqrt{a^2 - x^2}, x \in (0, a)$

Squaring on both sides, we get

$$z = A^2 = \frac{16b^2}{a^2}x^2(a^2 - x^2) = \frac{16b^2}{a^2}(a^2x^2 - x^4), x \in (0, a)$$

Here, A is maximum as z is maximum.

Now, differentiate w.r.t. x, we get

$$\frac{dz}{dx} = \frac{16b^2}{a^2}(2a^2x - 4x^3) = \frac{32b^2}{a^2} \cdot x(a^2 - 2x^2)$$

$$= \frac{32b^2}{a^2} \cdot x(a + \sqrt{2}x)(a - \sqrt{2}x)$$

For maximum or minimum of z, $\frac{dz}{dx} = 0$

$$\therefore \quad \frac{32b^2}{a^2} \cdot x(a + \sqrt{2}x)(a - \sqrt{2}x) = 0$$

$$\Rightarrow \quad x = \frac{a}{\sqrt{2}}$$

$$\left[\because x \text{ can't be zero or negative}, \therefore x \neq 0, -\frac{a}{\sqrt{2}}\right]$$

Again differentiate w.r.t. x, we get

$$\frac{d^2z}{dx^2} = \frac{32b^2}{a^2}(a^2 - 6x^2)$$

$$\therefore \text{ At } x = \frac{a}{\sqrt{2}}, \frac{d^2z}{dx^2} = \frac{32b^2}{a^2}\left(a^2 - 6 \times \frac{a^2}{2}\right)$$

$$= \frac{32b^2}{a^2} \times (-2a^2) = -64b^2 < 0$$

Hence, by using second derivative test, there is a local maximum value of z at the critical point $x = \frac{a}{\sqrt{2}}$. Since, there is only one critical point, therefore z is maximum at $x = \frac{a}{\sqrt{2}}$, hence A is maximum at $x = \frac{a}{\sqrt{2}}$.

Thus, for maximum area of the soccer field,

length of the soccer field $= 2x = 2 \times \frac{a}{\sqrt{2}} = a\sqrt{2}$

and width of the soccer field $= 2y = \frac{2b}{a}\sqrt{a^2 - x^2}$

$$= \frac{2b}{a}\sqrt{a^2 - \frac{a^2}{2}}$$

$$= \frac{2b}{a} \times \frac{a}{\sqrt{2}} = b\sqrt{2}$$

Case Study 9

The temperature of a person during an intestinal illness is given by $f(x) = -0.1x^2 + mx + 98.6$, $0 \le x \le 12$, m being a constant, where $f(x)$ is the temperature in °F at x days.

Based on the above information, solve the following questions: (CBSE SQP 2022-23)

Q 1. Is the function differentiable in the interval (0, 12)? Justify your answer.

Q 2. If 6 is the critical point of the function, then find the value of the constant m.

Q 3. Find the intervals in which the function is strictly increasing/strictly decreasing.

Or

Find the points of local maximum/local minimum, if any, in the interval (0, 12) as well as the points of absolute maximum/absolute minimum in the interval [0, 12]. Also, find the corresponding local maximum/local minimum and the absolute maximum/absolute minimum values of the function.

Solutions

1. Given, function $f(x) = -0.1x^2 + mx + 98.6$

   Here, $f(x)$ being a polynomial function. So, it is differentiable everywhere.

   Hence, function $f(x)$ is differentiable in the interval (0, 12).

2. Now, $f'(x) = -0.2x + m$

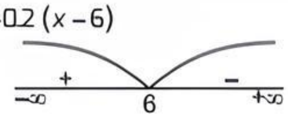
   Since, $x = 6$ is the critical point of the function.

   $\therefore \quad f'(6) = 0 \Rightarrow -0.2 \times 6 + m = 0 \Rightarrow m = 1.2$

3. Given, $f(x) = -0.1x^2 + mx + 98.6$, $x \in [0, 12]$

   $\Rightarrow \quad f'(x) = -0.2x + m = -0.2x + 1.2 \qquad [\because m = 1.2]$

   $= -0.2\,(x - 6)$



Intervals	Sign of $f'(x)$	Conclusion
$(-\infty, 6)$	+	f is strictly increasing in (0, 6)
(6, 12)	–	f is strictly decreasing in (6, 12)

Hence, $f(x)$ is strictly increasing in (0, 6) and strictly decreasing in (6, 12) because $x \in [0, 12]$.

Or

Given, $f(x) = -0.1x^2 + mx + 98.6$, $x \in [0, 12]$

$\Rightarrow \quad f(x) = -0.1x^2 + 1.2x + 98.6 \qquad [\because m = 1.2]$

Differentiate both sides w.r.t. 'x', we get

$f'(x) = -0.2x + 1.2 = 0.2\,(-x + 6)$

For maximum or minimum of $f(x)$,

$f'(x) = 0.2\,(-x + 6) = 0$

$\Rightarrow \quad x = 6$

Again differentiate both sides w.r.t. 'x', we get

$f''(x) = -0.2$

At $x = 6$, $\quad f''(6) = -0.2 < 0$

Hence, by second derivative test, $f(x)$ is maximum at $x = 6$. So, $x = 6$ is a point of local maximum in (0, 12).

Now, local maximum value $= f(6)$

$= -0.1\,(6)^2 + 1.2\,(6) + 98.6$

$= -0.1 \times 36 + 1.2 \times 6 + 98.6$

$= -3.6 + 7.2 + 98.6 = 102.2$

At $x = 0$, $\quad f(0) = -0.1 \times 0 + 1.2 \times 0 + 98.6$

$= 98.6$

At $x = 6$, $\quad f(6) = 102.2$

At $x = 12$, $\quad f(12) = -0.1 \times (12)^2 + 1.2 \times 12 + 98.6$

$= -14.4 + 14.4 + 98.6 = 98.6$

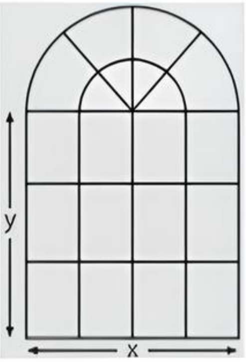
So, $x = 6$ is the point of absolute maximum and $x = 0, 12$ are the points of absolute minimum in the interval [0, 12].

Now, absolute maximum value $= f(6) = 102.2$

and absolute minimum value $= f(0)$ or $f(12) = 98.6$.

Case Study 10

Rohan, a student of class XII, visited his uncles's flat with his father. He observe that the window of the house is in the form of a rectangle surmounted by a semicircular opening having perimeter 10 m as shown in the figure.



Based on the given information, solve the following questions:

Q 1. If x and y represents the length and breadth of the rectangular region, then find the area (A) of the window in terms of x.

Q 2. Rohan is interested in maximising the area of the whole window, for this to happen, find the value of x.

Q 3. Find the maximum area of the window.

Or

For maximum value of A, find the breadth of rectangular part of the window.

Solutions

1. Given, perimeter of window = 10 m

$\therefore\quad x + y + y +$ circumference of semi-circle $= 10$

$$\Rightarrow \quad x + 2y + \pi\frac{x}{2} = 10 \quad \ldots(1)$$

So, area of window (A) = Area of rectangle + Area of semi-circle

$$\Rightarrow \quad A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$= x\left(5 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{2}\cdot\frac{\pi x^2}{4}$$

$$\left[\because \text{From eq. (1), } y = 5 - \frac{x}{2} - \frac{\pi x}{4}\right]$$

$$= 5x - \frac{x^2}{2} - \frac{\pi x^2}{4} + \frac{\pi x^2}{8} = 5x - \frac{x^2}{2} - \frac{\pi x^2}{8}$$

2. We have, $A = 5x - \frac{x^2}{2} - \frac{\pi x^2}{8}$

Differentiate w.r.t. x, we get

$$\frac{dA}{dx} = 5 - x - \frac{\pi x}{4}$$

For maximum or minimum of A,

$$\frac{dA}{dx} = 0 \quad \Rightarrow \quad 5 = x + \frac{\pi x}{4}$$

$$\Rightarrow \quad x(4+\pi) = 20 \quad \Rightarrow \quad x = \frac{20}{4+\pi}$$

Clearly, $\frac{d^2A}{dx^2} < 0$ at $x = \frac{20}{4+\pi}$

Hence, required value of x is $\frac{20}{4+\pi}$ m.

3. At $x = \frac{20}{4+\pi}$,

$$A = 5\left(\frac{20}{4+\pi}\right) - \left(\frac{20}{4+\pi}\right)^2\frac{1}{2} - \frac{\pi}{8}\left(\frac{20}{4+\pi}\right)^2$$

$$= \frac{100}{4+\pi} - \frac{200}{(4+\pi)^2} - \frac{50\pi}{(4+\pi)^2}$$

$$= \frac{(4+\pi)(100) - 200 - 50\pi}{(4+\pi)^2}$$

$$= \frac{400 + 100\pi - 200 - 50\pi}{(4+\pi)^2}$$

$$= \frac{200 + 50\pi}{(4+\pi)^2} = \frac{50(4+\pi)}{(4+\pi)^2} = \frac{50}{4+\pi}\text{ m}^2$$

Or

Since, area 'A' is maximum at $= \frac{20}{4+\pi}$.

$$\text{Now,}\quad y = 5 - \frac{x}{2} - \frac{\pi x}{4} = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

$$= 5 - x\left(\frac{2+\pi}{4}\right) = 5 - \left(\frac{20}{4+\pi}\right)\left(\frac{2+\pi}{4}\right)$$

$$= 5 - 5\frac{(2+\pi)}{4+\pi} = \frac{20 + 5\pi - 10 - 5\pi}{4+\pi}$$

$$= \frac{10}{4+\pi}\text{ m}$$

Case Study 11

An open water tank of aluminium sheet of negligible thickness, with a square base and vertical sides, is to be constructed in a farm for irrigation. It should hold 32000 L of water, that comes out from a tube well.

Based on the above information, solve the following questions:

Q 1. If the length, width and height of the open tank be x, x and y m respectively, then find the outer surface area of tank in terms of x.

Q 2. Show that the cost of material will be least when width of tank is equal to twice of its depth.

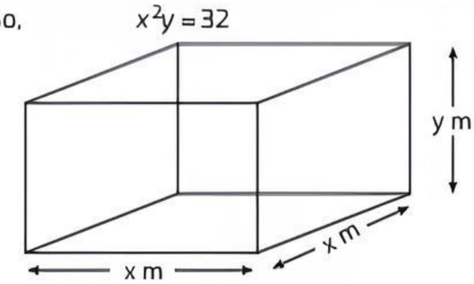
Q 3. If cost of aluminium sheet is ₹ 360/m^2, then find the minimum cost for the construction of tank.

Solutions

1. Since, volume of tank should be 32000 L.

$\therefore\quad x^2y\text{ m}^3 = 32000\text{ L} = 32\text{ m}^3 \quad [\because 1\text{ litre} = 0.001\text{ m}^3]$

$$\text{So,}\quad x^2y = 32 \quad \ldots(1)$$



Since, the tank is open from the top, therefore the surface area

$$= [x \times x + 2(xy + yx)]$$
$$= [x^2 + 2(2xy)]$$
$$= (x^2 + 4xy)\text{ m}^2$$

Let S be the outer surface area of tank.

Then, $S = x^2 + 4xy$

$\Rightarrow \quad S(x) = x^2 + 4x \cdot \frac{32}{x^2} = x^2 + \frac{128}{x} \quad [\because x^2y = 32] \quad ...(2)$

2. Differentiate eq. (2) w.r.t. x on both sides, we get

$$\frac{dS}{dx} = 2x - \frac{128}{x^2}$$

and $\frac{d^2S}{dx^2} = 2 + \frac{256}{x^3}$

For maximum or minimum values of S, put $\frac{dS}{dx} = 0$

$\therefore \quad 2x = \frac{128}{x^2}$

$\Rightarrow \quad x^3 = 64$

$\Rightarrow \quad x = 4$ m

At $x = 4$, $\frac{d^2S}{dx^2} = 2 + \frac{256}{4^3}$

$= 2 + 4 = 6 > 0$

$\therefore$ S is minimum when $x = 4$

Now as $x^2y = 32$, therefore $y = 2$

Thus, $x = 2y$

Since, surface area is minimum when $x = 2y$, therefore cost of material will be least when $x = 2y$.

Thus, cost of material will be least when width is equal to twice of its depth. **Hence proved.**

3. Since, minimum surface area

$= x^2 + 4xy = 4^2 + 4 \times 4 \times 2 = 48 \text{ m}^2$

and cost per m^2 = ₹ 360

$\therefore$ Minimum cost = ₹ (48×360) = ₹ 17280

Case Study 12

Sooraj's father wants to construct a rectangular garden using a brick wall on one side of the garden and wire fencing for the other three sides as shown in the figure. He has 200 metres of fencing wire.

Based on the above information, solve the following questions: *(CBSE 2023)*

Q 1. Let 'x' metre denotes the length of the side of the garden perpendicular to the brick wall and 'y' metre denotes the length of the side parallel to the brick wall. Determine the relation representing the total length of fencing wire and also write $A(x)$, the area of the garden.

Q 2. Determine the maximum value of $A(x)$.

Solutions

1. Given, x and y are the length and breadth of the rectangular garden and length of wire is 200 m. The relation representing the given problem is

$$2x + y = 200$$

Now, area of the garden, $A(x) = xy$

$= x(200 - 2x)$

$= (200x - 2x^2) \text{ m}^2$

2. $\because A(x) = 200x - 2x^2$

Differentiate w.r.t. x, we get

$$A'(x) = 200 - 4x$$

For maxima and minima, put $A'(x) = 0$

$\therefore \quad 200 - 4x = 0 \Rightarrow x = 50$

Now, $A''(x) = -4 < 0 \; \forall \; x$, so A is maximum at $x = 50$.

Hence, maximum value of $A(x)$ is

$A(50) = 200 \times 50 - 2(50)^2$

$= 10000 - 5000 = 5000 \text{ m}^2$

Very Short Answer Type Questions

Q 1. A balloon which always remains spherical, has a variable radius. Find the rate at which its volume is increasing with radius when the later is 10 cm. *(NCERT EXERCISE)*

Q 2. The total revenue in rupees received from the scale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue, when $x = 5$, where by marginal revenue, we mean the rate of change of total revenue with respect to the number of items sold at an instant. *(NCERT EXERCISE)*

Q 3. A stone is dropped into a quiet lake and waves move in circles at a speed of 4 cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing? *(NCERT EXERCISE)*

Q 4. An edge of a variable cube is increasing at the rate of 3 cm/sec. How fast is the volume of cube is increasing when the edge is 10 cm long? *(NCERT EXERCISE)*

Q 5. Let I be an interval disjoint from $(-1, 1)$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is increasing on I. *(NCERT EXERCISE)*

Q 6. The total cost $C(x)$ associated with the production of x unit of an item is given by

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000.$$

Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output. *(NCERT EXERCISE; CBSE 2018)*

Q 7. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R. *(NCERT EXERCISE, CBSE 2017)*

Q 8. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R. *(CBSE 2017)*

Short Answer Type-I Questions

Q 1. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then find the rate of change of the slope of the curve when $x = 3$. *(NCERT EXEMPLAR, CBSE 2017)*

Q 2. The length x, of a rectangle is decreasing at the rate of 5 cm/min and the width y, is increasing at the rate of 4 cm/min. When $x = 8$ cm and $y = 6$ cm, find the rate of change of the area of the rectangle. *(CBSE 2017)*

Q 3. Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate. *(CBSE 2023)*

Q 4. The radius r of a right circular cylinder is increasing uniformly at the rate of 0.3 cm/s and its height h is decreasing at the rate of 0.4 cm/s. When $r = 3.5$ cm and $h = 7$ cm, find the rate of change of the curved surface area of the cylinder. $\left(\text{Use } \pi = \frac{22}{7}\right)$ *(CBSE 2017)*

Q 5. The radius r of the base of a right circular cone is decreasing at the rate of 2 cm/min and its height h is increasing at the rate of 3 cm/min. When $r = 3.5$ cm and $h = 6$ cm, find the rate of change of the volume of the cone. $\left(\text{Use } \pi = \frac{22}{7}\right)$ *(CBSE 2017)*

Q 6. A man 1.6 m tall walks at the rate of 0.3 m/sec away from a street light that is 4 m above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow lengthening? *(CBSE SQP 2022-23)*

Q 7. The volume of a cube is increasing at the rate of $9 \text{ cm}^3/\text{s}$. How fast is its surface area increasing when the length of an edge is 10 cm? *(NCERT EXERCISE, CBSE 2017)*

Q 8. Find the interval in which the function $f(x) = 2x^3 - 3x$ is strictly increasing. *(CBSE 2023)*

Q 9. Find the interval(s) in which the function $f: R \to R$ defined by $f(x) = xe^x$, is increasing. *(CBSE SQP 2023-24)*

Q 10. Determine for which values of x, the function $y = x^4 - \frac{4x^3}{3}$ is increasing and for which values, it is decreasing. *(NCERT EXEMPLAR)*

Q 11. Show that the function $f(x) = \sin x$ is:

(i) strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

(ii) strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

(iii) neither increasing nor decreasing in $(0, \pi)$. *(NCERT EXERCISE)*

Q 12. Check whether the function $f: R \to R$ defined by $f(x) = x^3 + x$, has any critical point(s) or not. If yes, then find the points. *(CBSE SQP 2023-24)*

Q 13. Find the maximum and minimum values of the function given by $f(x) = 5 + \sin 2x$. *(CBSE 2023)*

Q 14. If $f(x) = \frac{1}{4x^2 + 2x + 1}$; $x \in R$, then find the maximum value of $f(x)$. *(CBSE SQP 2022-23)*

Q 15. Find the maximum profit that a company can make, if the profit function is given by $P(x) = 72 + 42x - x^2$, where x is the number of units and P is the profit in rupees. *(CBSE SQP 2022-23)*

Short Answer Type-II Questions

Q 1. If the area of any circle is increasing at a uniform rate, then prove that the increase rate of its perimeter is inversely proportional to radius. *(NCERT EXEMPLAR)*

Q 2. A man of height 2 m walks at a uniform speed of 5 km/h away from a lamp post which is 6 m high. Find the rate at which the length of his shadow increases. *(NCERT EXERCISE)*

Q 3. Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of $2 \text{ cm}^2/\text{sec}$ in the surface area, through a tiny hole at the vertex of the bottom. When the slant height of the cone is 4 cm, find the rate of decrease of the slant height of water. *(NCERT EXEMPLAR)*

Q 4. Show that the function $f(x) = \tan x - 4x$ is strictly decreasing in the interval $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$. *(NCERT EXEMPLAR)*

Q 5. Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is:

(i) strictly increasing, (ii) strictly decreasing. *(CBSE 2018)*

Q 6. Find the intervals in which the function $f(x) = \sin 3x$, $x \in \left[0, \frac{\pi}{2}\right]$ is increasing or decreasing. *(NCERT EXERCISE)*

Q 7. Find two parts of 100 such that the sum of twice of one part and the square of other part is minimum.

Q 8. Find two positive numbers whose sum is 15 and the sum of whose squares is minimum. *(NCERT EXERCISE)*

Q 9. Show that the function $\left[x^2 \log \frac{1}{x}\right]$ is maximum at $x = \frac{1}{\sqrt{e}}$.

Q 10. Show that the value of the function x^x is minimum at $x = \frac{1}{e}$. *(NCERT EXEMPLAR)*

Q 11. Find the maximum value of $\frac{\log x}{x}$, where $0 < x < \infty$. *(NCERT EXERCISE)*

Q 12. Prove that the value of $\sin x\,(1 + \cos x)$ is maximum at $x = \frac{1}{3}\pi$. Also find its maximum value.

Q 13. Prove that the maximum value of $(\sin x + \cos x)$ is $\sqrt{2}$. *(NCERT EXEMPLAR)*

Q 14. Find the local maximum and local minimum values of the function $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$. *(NCERT EXERCISE)*

Q 15. If the value of $y = a \log x + bx^2 + x$ is maximum or minimum at $x = -1$ and $x = 2$, find the values of a and b.

Long Answer Type Questions

Q 1. The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing. *(CBSE 2023)*

Q 2. A water tank has the slope of an inverted right circular cone with its axis vertical and vertex lower most. Its semi-vertical angle is $\tan^{-1}\frac{1}{2}$. Water is poured into it at a constant rate of 5 cubic metre per minute. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m. *(NCERT EXERCISE)*

Q 3. A ladder 13 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 1.5 m/sec. How fast is the angle between ladder and ground is changing when the foot of the ladder is 12 m away from the wall? *(NCERT EXEMPLAR)*

Q 4. Find the intervals in which the function $f(x) = (x - 1)^3\ (x - 2)^2$ is strictly increasing or strictly decreasing. *(NCERT EXEMPLAR)*

Q 5. Find the intervals in which the function $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$ is strictly increasing or strictly decreasing. *(NCERT EXERCISE)*

Q 6. The sum of two numbers is given. Prove that their product will be maximum, if each one is half of their sum. *(NCERT EXERCISE)*

Q 7. Sum of two numbers is 5. If the sum of the cubes of these numbers is least, then find the sum of the squares of these numbers. *(CBSE 2023)*

Q 8. If $2x + y = 1$, then find the maximum value of the function x^2y.

Q 9. If $x + y = 10$, then find the maximum value of the function xy^2.

Q 10. Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, 1). *(CBSE 2020)*

Q 11. An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, place at (3, 7), wants to shoot down the helicopter when it is nearest to him. Find the nearest distance. *(NCERT EXERCISE)*

Q 12. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. *(CBSE 2017)*

Q 13. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base. *(NCERT EXERCISE)*

Q 14. Prove that the radius of the right circular cylinder of greatest curved surface area inscribed in a given cone is half of that of the cone. *(NCERT EXERCISE)*

Q 15. Prove that the least canvas is required to make a conical tent of given volume, if its height is $\sqrt{2}$ times the radius of the base.

Or

Show that a conical tent of given volume required least canvas if the ratio of its height and radius of base is $\sqrt{2} : 1$. *(NCERT EXERCISE)*

Q 16. AB is the diameter of a circle and C is any point on the circle. Show that the area of triangle ABC is maximum, when it is an isosceles triangle. *(NCERT EXEMPLAR, CBSE 2017)*

Q 17. A metal box with a square base and vertical sides is to contain 1024 cm^3. The material for the top and bottom costs ₹5 per cm^2 and the material for the sides costs ₹ 2.50 per cm^2. Find the least cost of the box. *(NCERT EXEMPLAR, CBSE 2017)*

Q 18. Show that the semi-vertical angle of cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$. *(NCERT EXERCISE)*

Q 19. If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$. *(NCERT EXEMPLAR, CBSE 2017)*

Q 20. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find the maximum volume of cone. *(CBSE 2019)*

Or

Prove that the ratio of the height of a cone of maximum volume inscribed in a given sphere to the radius of sphere is 4 : 3. *(NCERT EXERCISE)*

Q 21. A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a rectangle whose length is twice its breadth. What should be the lengths of the two pieces, so that the combined area of the square and the rectangle is minimum? *(CBSE 2017)*

Q 22. A window is in the form of a rectangle surmounted by a semi-circular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening. *(NCERT EXERCISE, CBSE 2017)*

Q 23. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square metre for the sides, what is the cost of least expensive tank? *(NCERT EXERCISE; CBSE 2019)*

Q 24. Show that the semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\frac{1}{3}$. *(NCERT EXERCISE)*

Solutions

Very Short Answer Type Questions

1. Let r be the radius and V be the volume of the balloon.

Then, $V = \frac{4}{3}\pi r^3$

TiP

Adequate practice is required in problems involving rate of change of bodies.

$\Rightarrow \quad \frac{dV}{dr} = \frac{4}{3}\pi \times 3r^2 = 4\pi r^2$

$\therefore \quad \left[\frac{dV}{dr}\right]_{r=10} = 4\pi \times 10^2$

$= 400\pi$ cubic cm/cm

Therefore, rate of change of volume of balloon with respect to radius is 400π cubic cm/cm when radius is 10 cm.

2. Given that, total revenue $R(x) = 3x^2 + 36x + 5$.

TR!CK

Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$\therefore$ Marginal revenue (MR) $= \frac{dR}{dx} = 6x + 36$

When $x = 5$, then

$MR = 6(5) + 36 = 30 + 36 = 66$

Hence, the required revenue is ₹ 66 per unit.

3. The area A of a circle with radius r is given by $A = \pi r^2$. Therefore, the rate of change of area A with respect to time t is

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2)\cdot\frac{dr}{dt} = 2\pi r\frac{dr}{dt}$$

[by chain rule]

It is given that, $\frac{dr}{dt} = 4$ cm/s

Therefore, when $r = 10$ cm, $\frac{dA}{dt} = 2\pi(10)(4) = 80\pi$

Thus, the enclosed area is increasing at the rate of 80π cm^2/s, when $r = 10$ cm.

4. Let at any time t, the edge of cube be x and volume be V.

Then, $V = x^3$

Given, $\frac{dx}{dt} = 3$ cm/sec

Now, $V = x^3$

$\Rightarrow \quad \frac{dV}{dt} = 3x^2 \times \frac{dx}{dt} = 3x^2 \times 3 = 9x^2$

$\Rightarrow \quad \left[\frac{dV}{dt}\right]_{x=10} = 9 \times 10^2$

$= 9 \times 100 = 900$ cubic cm/sec

Therefore, volume of cube is increasing at the rate of 900 cubic cm/sec when edge is 10 cm.

5. Given, $f(x) = x + \frac{1}{x}$

$\therefore \quad f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$

Now, $x \in I \Rightarrow x \notin (-1, 1)$

$\Rightarrow \quad x \le -1$ or $x \ge 1 \Rightarrow x^2 \ge 1$

$\Rightarrow \quad x^2 - 1 \ge 0$

$\Rightarrow \quad \frac{x^2 - 1}{x^2} \ge 0 \qquad [\because x^2 \ge 1 > 0]$

$\Rightarrow \quad f'(x) \ge 0$

So, for each $x \in I$, $f'(x) \ge 0$

Hence, $f(x)$ is increasing on I. **Hence proved.**

6. Given that,

Total cost, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$

TiP

Marginal cost is the rate of change of total cost with respect to output.

$\therefore$ Marginal cost $= \frac{dC}{dx} = 0.005(3x^2) - 0.02(2x) + 30$

At $x = 3$ units,

$MC = 0.005\{3(3)^2\} - 0.02(2 \times 3) + 30$

$= 0.005 \times 27 - 0.02 \times 6 + 30$

$= 0.135 - 0.12 + 30 = 30.135 - 0.12 =$ ₹ 30.015

7. Given, function $f(x) = x^3 - 3x^2 + 6x - 100$

Now, $f'(x) = 3x^2 - 6x + 6$

$\Rightarrow \quad f'(x) = 3(x^2 - 2x + 2)$

$\Rightarrow \quad f'(x) = 3\{(x-1)^2 + 1\}$

$\therefore \quad f'(x) > 0$ for all $x \in R$

So, $f(x)$ is increasing on R. **Hence proved.**

8. Given, function $f(x) = 4x^3 - 18x^2 + 27x - 7$

Now, $f'(x) = 12x^2 - 36x + 27$

$\Rightarrow f'(x) = 3(4x^2 - 12x + 9)$

$\Rightarrow f'(x) = 3(2x - 3)^2$

$\therefore f'(x) \geq 0$ for all $x \in R$

So, $f(x)$ is always increasing on R. **Hence proved.**

Short Answer Type-I Questions

1.

TiP

$\frac{dy}{dx}$ is positive if y increases as x increases and is negative if y decreases as x increases.

It is given that 'x' increases at the rate of 2 units/sec for the curve $y = 5x - 2x^3$.

$$\frac{dx}{dt} = 2 \text{ units/sec}$$

$\because \quad y = 5x - 2x^3$

Differentiate w.r.t. 'x' on both sides,

$$\frac{dy}{dx} = 5 \cdot 1 - 6x^2$$

Let $\quad m = \frac{dy}{dx} = 5 - 6x^2 \quad$ ...(1)

which is the slope of the curve.

Now, differentiate eq. (1) w.r.t. 't' on both sides, we have

$$\frac{dm}{dt} = 0 - 12x \cdot \frac{dx}{dt}$$

$$\therefore \left[\frac{dm}{dt}\right]_{x=3} = -12 \times 3 \times 2 = -72 \text{ units/sec}$$

$$\left[\because \frac{dx}{dt} = 2 \text{ unit/sec}\right]$$

Hence, the slope of the curve is decreasing at the rate of 72 units/sec.

2. It is given that length (x) is decreasing at the rate of 5 cm/min and the width (y) is increasing at the rate of 4 cm/min.

TiP

$\frac{dy}{dx}$ is positive if y increases as x increases and is negative if y decreases as x increases.

$$\Rightarrow \frac{dx}{dt} = -5 \text{ cm/min} \quad \text{and} \quad \frac{dy}{dt} = 4 \text{ cm/min.}$$

Thus, the area (A) of a rectangle is, $A = x \cdot y$

Differentiate w.r.t. 't', we get

$$\frac{dA}{dt} = x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}$$

$$\therefore \left[\frac{dA}{dt}\right]_{x=8,\, y=6} = 8(4) + 6(-5)$$

$$= 32 - 30 = 2 \text{ cm}^2/\text{min}$$

Hence, the area is increasing at the rate of 2 cm^2/min.

3. Let required point on the curve $y^2 = 8x$ be (x, y).

According to the given condition,

$$\frac{dy}{dt} = \frac{dx}{dt} \quad \text{...(1)}$$

Consider, $\quad y^2 = 8x$

Differentiate w.r.t. t, we get

$$2y\frac{dy}{dt} = \frac{8dx}{dt}$$

$$\Rightarrow 2y = 8 \quad \left[\because \frac{dy}{dt} = \frac{dx}{dt}\right]$$

$$\Rightarrow y = 4$$

Put $y = 4$ in $y^2 = 8x$, we get

$$4^2 = 8x \Rightarrow x = 2$$

Hence, required point is (2, 4).

4. Let the curved surface area of right circular cylinder be S at any time t.

Given; the radius 'r' of a right circular cylinder is increasing uniformly at the rate of 0.3 cm/s.

i.e., $\quad \frac{dr}{dt} = 0.3$ cm/s

and the height 'h' is decreasing at the rate of 0.4 cm/s.

i.e., $\quad \frac{dh}{dt} = -0.4$ cm/s

$\because \quad S = 2\pi rh$

Differentiate w.r.t. 't' on both sides, we get

$$\frac{dS}{dt} = 2\pi\frac{d}{dt}(rh)$$

$$\Rightarrow \frac{dS}{dt} = 2\pi\left\{r \cdot \frac{dh}{dt} + h \cdot \frac{dr}{dt}\right\}$$

TR!CK

$\frac{dy}{dx}$ is positive if y increases as x increases and is negative if y decreases as x increases.

$$= 2\pi\{r(-0.4) + h(0.3)\} = 2\pi\{0.3h - 0.4r\}$$

Now, $$\left[\frac{dS}{dt}\right]_{(\text{at } r = 3.5 \text{ cm},\ h = 7 \text{ cm})} = 2\pi\{0.3(7) - 0.4(3.5)\}$$

$$= 2\pi\{2.1 - 1.4\} = 2 \times \frac{22}{7} \times 0.7 = 4.4 \text{ cm}^2/\text{s}$$

Hence, the rate of change of the curved surface area of the cylinder is 4.4 cm^2/s.

5. Let the volume of right circular cone be V at any time t.

Given, the radius 'r' of the base of a right circular cone is decreasing at the rate of 2 cm/min.

i.e., $\quad \frac{dr}{dt} = -2$ cm/min.

TR!CK

$\frac{dy}{dx}$ is positive if y increases as x increases and is negative if y decreases as x increases.

and the height 'h' is increasing at the rate of 3 cm/min.

i.e., $\quad \frac{dh}{dt} = 3$ cm/min

$\because \quad V = \frac{1}{3}\pi r^2 h$

Differentiate w.r.t. 't' on both sides, we get

$$\frac{dV}{dt} = \frac{1}{3}\pi \frac{d}{dt}(r^2h) = \frac{\pi}{3}\left\{r^2\frac{dh}{dt} + 2r\frac{dr}{dt}\times h\right\}$$

$$\therefore \frac{dV}{dt} = \frac{\pi}{3}\{r^2(3) + 2r(-2)h\} = \frac{\pi}{3}(3r^2 - 4rh)$$

Now, $\left[\frac{dV}{dt}\right]_{\left(\begin{smallmatrix}\text{at } r = 3.5 \text{ cm}\\ h = 6 \text{ cm}\end{smallmatrix}\right)} = \frac{\pi}{3}\{3(3.5)^2 - 4\times 3.5\times 6\}$

$$= \frac{22}{3\times 7}\{36.75 - 84\} = \frac{22}{3\times 7}(-47.25)$$

$$= -22\times 2.25 = -49.5 \text{ cm}^3/\text{min}$$

Hence, the volume of the cone is decreasing at the rate of 49.5 cm^3/min.

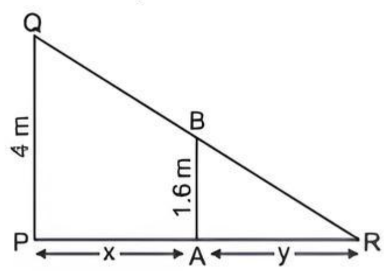
6. Let $PQ = 4$ m be the height of pole and $AB = 1.6$ m be the height of man.

Let the end of shadow be R and it is at a distance of y metre from A when the man is at a distance x metre from PQ at some instant.

Since, ΔPQR and ΔABR are similar.

We have, $\frac{PQ}{AB} = \frac{PR}{AR}$

$\Rightarrow \quad \frac{4}{1.6} = \frac{x+y}{y} \quad \Rightarrow \quad 2x = 3y$



$\Rightarrow \quad 2\frac{dx}{dt} = 3\frac{dy}{dt}$

(differentiating both sides w.r.t. 't')

$\Rightarrow \quad \frac{dy}{dt} = \frac{2}{3}\times\frac{dx}{dt} = \frac{2}{3}\times 0.3 = 0.2$

$\left[\because \frac{dx}{dt} = 0.3 \text{ m/s (given)}\right]$

So, the rate at which the tip of his shadow moving

$$= \left(\frac{dx}{dt} + \frac{dy}{dt}\right) \text{m/s} = (0.3 + 0.2) = 0.5 \text{ m/s}$$

and the rate at which his shadow is lengthening

$$= \frac{dy}{dt} = 0.2 \text{ m/s}$$

7. Let the length of edge of cube be x, total surface area S and volume V at any time t.

Then $\quad V = x^3$ and $S = 6x^2$

Given that, $\frac{dV}{dt} = 9 \text{ cm}^3/\text{s}$

$\therefore \quad \frac{d}{dt}(x^3) = 9$

$\Rightarrow \quad 3x^2\frac{dx}{dt} = 9$

$\Rightarrow \quad \frac{dx}{dt} = \frac{9}{3x^2} = \frac{3}{x^2}$

TiP

$\frac{dy}{dx}$ is positive, if y increases as x increases and it is negative, if y decreases as x increases.

Now, $\quad S = 6x^2$

$\Rightarrow \quad \frac{dS}{dt} = 6\times 2x\times\frac{dx}{dt}$

$\Rightarrow \quad \frac{dS}{dt} = 12x\times\frac{3}{x^2} = \frac{36}{x}$

$\therefore \quad \left[\frac{dS}{dt}\right]_{x=10} = \frac{36}{10} = 3.6 \text{ cm}^2/\text{s}$

Hence, the surface area of cube is increasing at the rate of 3.6 cm^2/s. when length of its side is 10 cm.

8. Given, $f(x) = 2x^3 - 3x$

Differentiate w.r.t. x, we get

$$f'(x) = 6x^2 - 3$$

For $f(x)$ to be strictly increasing,

$$f'(x) > 0$$

$\therefore \; 6x^2 - 3 > 0 \Rightarrow x^2 > \frac{1}{2}$

$\Rightarrow \quad x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right)\cup\left(\frac{1}{\sqrt{2}}, \infty\right)$

9. Given, $f(x) = xe^x$

Differentiate both sides w.r.t. x, we get

$f'(x) = xe^x + e^x$

For $f(x)$ to be increasing, $f'(x)\ge 0 \Rightarrow e^x(x+1)\ge 0$

$\Rightarrow \quad x + 1 \ge 0 \qquad [\because e^x > 0]$

$\Rightarrow \quad x \ge -1$

Hence, $f(x)$ is increasing in the interval $[-1, \infty)$.

10. Given, $y = x^4 - \frac{4x^3}{3}$

TiP

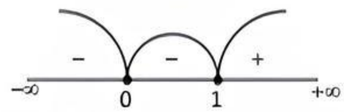
Practice more problems on finding increasing/ decreasing intervals.

Differentiate both sides with respect to x, we get

$$\frac{dy}{dx} = 4x^3 - 4x^2 = 4x^2(x-1)$$

Now, $\quad \frac{dy}{dx} = 0 \Rightarrow 4x^2(x-1)$

$\Rightarrow \quad x = 0, x = 1$



Since $f'(x) \le 0 \;\forall\; x \in (-\infty, 0]\cup[0, 1]$ and $f'(x) > 0$ $\forall\; x \in [1, \infty)$, so f is decreasing in $(-\infty, 1]$ and f is increasing in $[1, \infty)$.

11. Given, $\quad f(x) = \sin x$

$\Rightarrow \quad f'(x) = \cos x$

(i) $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \cos x > 0 \Rightarrow f'(x) > 0$

So, for each $x \in \left(0, \frac{\pi}{2}\right)$, $f'(x) > 0$

Thus, $f(x)$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

Hence proved.

(ii) $x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \cos x < 0 \Rightarrow f'(x) < 0$

So, for each $x \in \left(\frac{\pi}{2}, \pi\right)$, $f'(x) < 0$

Thus, $f(x)$ is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

Hence proved.

(iii) In the above parts (i) and (ii), we have shown that, The function $f(x) = \sin x$ is strictly increasing in the interval $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing in the interval $\left(\frac{\pi}{2}, \pi\right)$.

Therefore, function $f(x)$ is neither increasing nor decreasing in the interval $(0, \pi)$. **Hence proved.**

12.

TiP

Critical points are these points in which the derivative is zero.

Given, $f(x) = x^3 + x$

Differentiate both sides w.r.t. x, we get

$$f'(x) = 3x^2 + 1$$

For finding critical points, put

$$3x^2 + 1 = 0$$

$\Rightarrow$ $x^2 = -\frac{1}{3}$, which is not possible.

Hence, no critical point exist.

13. Given, $f(x) = 5 + \sin 2x$

Differentiate w.r.t. x, we get

$$f'(x) = 0 + 2\cos 2x$$

and $f''(x) = -4\sin 2x$

For maxima and minima, put $f'(x) = 0$

$\therefore$ $2\cos 2x = 0 \Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

or $x = \frac{\pi}{4}, \frac{3\pi}{4}$

When $x = \frac{\pi}{4}$,

$$f''\left(\frac{\pi}{4}\right) = -4\sin 2 \times \frac{\pi}{4} = -4 < 0$$

Thus, $f(x)$ is maximum at $x = \frac{\pi}{4}$.

$\therefore$ The maximum value is

$$f\left(\frac{\pi}{4}\right) = 5 + \sin 2 \times \frac{\pi}{4} = 5 + \sin\frac{\pi}{2}$$
$$= 5 + 1 = 6$$

When $x = \frac{3\pi}{4}$,

$$f''\left(\frac{3\pi}{4}\right) = -4\sin 2 \times \frac{3\pi}{4} = -4 \times (-1) = 4 > 0$$

Thus, $f(x)$ is minimum at $x = \frac{3\pi}{4}$.

$\therefore$ The minimum value is

$$f\left(\frac{3\pi}{4}\right) = 5 + \sin 2 \times \frac{3\pi}{4} = 5 - 1 = 4$$

14. For f to be maximum, $4x^2 + 2x + 1$ Should be minimum i.e.,

$$4x^2 + 2x + 1 = 4\left(x + \frac{1}{4}\right)^2 + \left(1 - \frac{1}{4}\right)$$
$$= 4\left(x + \frac{1}{4}\right)^2 + \frac{3}{4}$$

So, the minimum value of $4x^2 + 2x + 1$ is $\frac{3}{4}$.

Hence, maximum value of $f = \frac{1}{\left(\frac{3}{4}\right)} = \frac{4}{3}$.

15. Given, $P(x) = 72 + 42x - x^2$

Differentiate w.r.t. x, we get

$$P'(x) = 42 - 2x$$

For maximum or minimum profit, put $P'(x) = 0$

$\therefore$ $42 - 2x = 0 \Rightarrow x = 21$

Now $P''(x) = -2$

At $x = 21$, $P''(x) < 0$, which is maximum.

$\therefore$ Maximum profit at $x = 21$ is

$P(21) = 72 + 42(21) - (21)^2 = 72 + 882 - 441 =$ ₹ 513

Short Answer Type-II Questions

1. Let the area of the circle, $A = \pi r^2$

Differentiate both sides w.r.t. t,

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad ...(1)$$

TiP

Adequate practice on problems based on applications of derivatives.

and the perimeter of the circle,

$$P = 2\pi r$$

Differentiate both sides w.r.t. t,

$$\frac{dP}{dt} = 2\pi\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{2\pi}\frac{dP}{dt}$$

Put the value of $\frac{dr}{dt}$ in eq. (1),

$$\frac{dA}{dt} = 2\pi r \cdot \frac{1}{2\pi} \cdot \frac{dP}{dt} = r \cdot \frac{dP}{dt}$$

$$\Rightarrow \frac{dP}{dt} = \frac{1}{r} \cdot \frac{dA}{dt}$$

But given that, the area of any circle is increasing at a uniform rate.

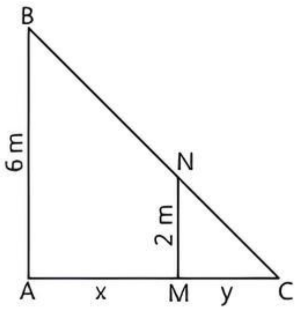
$$\therefore \frac{dA}{dt} = k \Rightarrow \frac{dP}{dt} = \frac{1}{r} \cdot k = \frac{k}{r}$$

$$\Rightarrow \frac{dP}{dt} \propto \frac{1}{r}$$

Hence, the increase rate of perimeter is inversely proportional to radius. **Hence proved.**

2. Let AB be the lamp post. Let at any time t, the man MN is x metre away from the lamp post.

Let the shadow MC of man is of length y metre.



Given, $\frac{dx}{dt} = 5$ km/h

Clearly, triangles ABC and MNC are similar.

So, $\frac{AB}{MN} = \frac{AC}{MC}$

$\Rightarrow \frac{6}{2} = \frac{x+y}{y} \Rightarrow \frac{x+y}{y} = 3$

$\Rightarrow x + y = 3y$

$\Rightarrow 2y = x$

Differentiate w.r.t. 't', we get

$$2 \times \frac{dy}{dt} = \frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2} \times \frac{dx}{dt} = \frac{1}{2} \times 5 = 2.5 \text{ km/h}$$

Hence, the rate of change of length of the shadow of man is 2.5 km/h.

3. Let S represents the surface area of conical funnel.

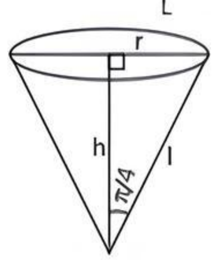
Then, $\frac{dS}{dt} = 2 \text{ cm}^2/\text{sec}$

TR!CK

$\frac{dy}{dx}$ is positive, if y increases as x increases and it is negative, if y decreases as x increases.

$\therefore \quad S = \pi r l = \pi \left(l \sin\frac{\pi}{4}\right) l = \frac{\pi}{\sqrt{2}} l^2$

$\left[\because \sin\frac{\pi}{4} = \frac{r}{l} \Rightarrow r = l \sin\frac{\pi}{4}\right]$



$$\Rightarrow \frac{dS}{dt} = \frac{2\pi}{\sqrt{2}} l \frac{dl}{dt} = \sqrt{2}\,\pi l \frac{dl}{dt}$$

When $l = 4$ cm,

$$\frac{dl}{dt} = \frac{1}{\sqrt{2}\pi l} \cdot \frac{dS}{dt} = \frac{1}{\sqrt{2}\pi(4)} \cdot 2 = \frac{1}{2\sqrt{2}\pi}$$

$$= \frac{\sqrt{2}}{4\pi} \text{ cm/sec}$$

Therefore, the rate of decrease of slant height is $\frac{\sqrt{2}}{4\pi}$ cm/sec.

4. Given, $f(x) = \tan x - 4x$

$$\therefore \quad f'(x) = \sec^2 x - 4 = \frac{1}{\cos^2 x} - 4 = \frac{1 - 4\cos^2 x}{\cos^2 x}$$

$$= \frac{4}{\cos^2 x}\left[\frac{1}{4} - \cos^2 x\right]$$

$[\because a^2 - b^2 = (a-b)(a+b)]$

$$= (4\sec^2 x)\left[\frac{1}{2} - \cos x\right]\left[\frac{1}{2} + \cos x\right]$$

Now, $x \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$

$\Rightarrow \frac{-\pi}{3} < x < \frac{\pi}{3} \Rightarrow \cos x > \frac{1}{2}$

$\Rightarrow \sec x < 2$

$\Rightarrow 4\sec^2 x > 0, \frac{1}{2} - \cos x < 0$ and $\frac{1}{2} + \cos x > 0$

$\Rightarrow (4\sec^2 x)\left[\frac{1}{2} - \cos x\right]\left[\frac{1}{2} + \cos x\right] < 0$

$\Rightarrow f'(x) < 0$

Thus, for each $x \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, $f'(x) < 0$

Hence, $f(x)$ is strictly decreasing in the interval $\left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$.

Hence proved.

5. We have, $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$

Differentiate w.r.t. 'x' on both sides, we get

$$f'(x) = \frac{4x^3}{4} - 3x^2 - 10x + 24$$

$$= x^3 - 3x^2 - 10x + 24$$

$$= x^3 - 2x^2 - x^2 + 2x - 12x + 24$$

$$= x^2(x-2) - x(x-2) - 12(x-2)$$

$$= (x-2)(x^2 - x - 12)$$

TR!CK

$\because \; 12 = 2 \times 6 = 4 \times 3 = 12 \times 1$

Here we will take 4 and 3 as a factors of 12.

So, the middle term become $-1 = 3 - 4$.

$$= (x-2)\{x^2 - 4x + 3x - 12\}$$

$$= (x-2)\{x(x-4) + 3(x-4)\}$$

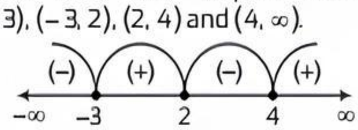
$$= (x-4)(x-2)(x+3)$$

TR!CKS

- *If $x_1 < x_2$ in $I \Rightarrow f(x_1) < f(x_2)$, $\forall\, x_1, x_2 \in I$*
  $\Rightarrow$ Strictly increasing on I.
- *If $x_1 < x_2$ in $I \Rightarrow f(x_1) > f(x_2)$, $\forall\, x_1, x_2 \in I$*
  $\Rightarrow$ Strictly decreasing on I.

Now, put $f'(x) = 0$, which gives $x = -3, 2$ and 4.

The points $x = -3$, $x = 2$ and $x = 4$ divides the whole real line into four disjoint intervals namely, $(-\infty, -3)$, $(-3, 2)$, $(2, 4)$ and $(4, \infty)$.



Here, $f'(x) > 0$ in the intervals $(-3, 2)$ and $(4, \infty)$

$f'(x) < 0$ in the intervals $(-\infty, -3)$ and $(2, 4)$.

So, $f(x)$ is strictly increasing in $(-3, 2) \cup (4, \infty)$, strictly decreasing in $(-\infty, -3) \cup (2, 4)$.

6. Given, $f(x) = \sin 3x,\ x \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \quad f'(x) = 3\cos 3x$

Now, $x \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \quad 0 \le x \le \frac{\pi}{2} \quad \Rightarrow \quad 0 \le 3x \le \frac{3\pi}{2}$

The function $f(x)$ will be increasing if $f'(x) \ge 0$.

Now, $f'(x) \ge 0 \Rightarrow 3\cos 3x \ge 0 \Rightarrow \cos 3x \ge 0$

$\Rightarrow \quad 0 \le 3x \le \frac{\pi}{2} \quad \Rightarrow \quad 0 \le x \le \frac{\pi}{6} \Rightarrow x \in \left[0, \frac{\pi}{6}\right]$

Therefore, $f(x)$ is increasing in the interval $\left[0, \frac{\pi}{6}\right]$.

The function $f(x)$ will be decreasing if $f'(x) \le 0$.

Now, $f'(x) \le 0 \Rightarrow 3\cos 3x \le 0 \Rightarrow \cos 3x \le 0$

$\Rightarrow \quad \frac{\pi}{2} \le 3x \le \frac{3\pi}{2} \quad \Rightarrow \quad \frac{\pi}{6} \le x \le \frac{\pi}{2} \Rightarrow x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

Therefore, $f(x)$ is decreasing in the interval $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

7. Let one part of 100 be x.

$\therefore$ Other part will be $(100 - x)$

According to the question, let $y = 2x + (100 - x)^2$

Differentiate both sides w.r.t. x,

$$\frac{dy}{dx} = 2 + 2(100 - x)(0 - 1) = -198 + 2x$$

For maximum and minimum value of y, put $\frac{dy}{dx} = 0$

$\Rightarrow \quad -198 + 2x = 0 \Rightarrow x = \frac{198}{2} = 99$

and $\quad \frac{d^2y}{dx^2} = 0 + 2 = \text{positive}$

At $x = 99$, y is minimum.

Therefore, one part is 99 and second part

$= 100 - 99 = 1$.

8. Let the first number be x, then second number will be $(15 - x)$.

Let sum of squares of these numbers be $S(x)$, then

$$S(x) = x^2 + (15 - x)^2 = 2x^2 - 30x + 225$$

$\Rightarrow \quad S'(x) = 4x - 30 \Rightarrow S''(x) = 4$

For minimum or maximum of S, put $S'(x) = 0$

$\Rightarrow \quad 4x - 30 = 0 \Rightarrow x = \frac{15}{2}$

and $\quad S''\left(\frac{15}{2}\right) = 4 > 0$

$\therefore$ Local minimum point of S is $x = \frac{15}{2}$ by second differential test. So, the sum of the squares of these numbers will be minimum, when the numbers are $\frac{15}{2}$ and $15 - \frac{15}{2} = \frac{15}{2}$.

9. Let $\quad y = x^2 \log \frac{1}{x}$

$\Rightarrow \quad y = x^2 \log x^{-1} = -x^2 \log x$

Differentiate both sides w.r.t. x,

$$\frac{dy}{dx} = -\left[x^2 \cdot \frac{1}{x} + \log x\,(2x)\right]$$

$$= -[x + 2x \log x]$$

For maximum or minimum value of y, put $\frac{dy}{dx} = 0$

$\Rightarrow \quad -[x + 2x \log x] = 0$

$\Rightarrow \quad x(1 + 2\log x) = 0$

$\Rightarrow \quad x = 0$

or $\quad 1 + 2\log_e x = 0$

$\Rightarrow \quad \log_e x = -\frac{1}{2}$

$\Rightarrow \quad x = e^{-\frac{1}{2}}$

Again differentiate with respect to x,

$$\frac{d^2y}{dx^2} = -\left[1 + 2x \cdot \frac{1}{x} + (\log x) \cdot 2\right]$$

$$= -[1 + 2 + 2\log x] = -[3 + 2\log x]$$

Put $\quad x = \frac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$,

$$\frac{d^2y}{dx^2} = -\left[3 + 2\log e^{-\frac{1}{2}}\right] = -\left[3 - 2 \cdot \frac{1}{2}\log e\right]$$

$$= -[3 - 1] = -2 \text{ [Negative]}$$

Therefore, at $x = \frac{1}{\sqrt{e}}$, function is maximum.

Hence proved.

10. Let $\quad y = x^x$

$\because \quad \log y = x \log x$

Differentiate both sides with respect to x,

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \cdot \log x$$

or $\quad \frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x) \quad \ldots(1)$

Again, differentiate with respect to x,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\{x^x(1 + \log x)\}$$

$$= x^x \cdot \frac{1}{x} + (1 + \log x) \cdot \frac{d}{dx}(x^x)$$

$$= x^x \cdot \left(\frac{1}{x}\right) + (1 + \log x) \cdot x^x (1 + \log x)$$

$$= x^x \{(1/x) + (1 + \log x)^2\}$$

For maxima/minima of y, put $\frac{dy}{dx} = 0$

$\Rightarrow \quad x^x(1 + \log x) = 0$

$\Rightarrow \quad 1 + \log x = 0$

or $\quad \log x = -1 = -\log e = \log\left(\frac{1}{e}\right) \Rightarrow x = \frac{1}{e}$

When $x = 1/e$, then $\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{1/e} \cdot e = \text{positive}$.

Therefore at $x = 1/e$, y is minimum and minimum value $= \left(\frac{1}{e}\right)^{1/e}$.

Hence proved.

COMMON ERR(!)R

Sometimes students forget to find the second derivative of the given function which leads to incorrect result.

11. Let $y = \frac{\log x}{x}$

Differentiate both sides with respect to x.

$$\frac{dy}{dx} = y_1 = \frac{x \cdot (1/x) - (\log x) \cdot 1}{x^2} = \frac{1 - \log x}{x^2}$$

and $$\frac{d^2y}{dx^2} = y_2 = \frac{x^2(-1/x) - (1 - \log x) \cdot 2x}{x^4}$$

$$= \frac{2\log x - 3}{x^3}$$

For maxima/minima of y, put $y_1 = 0$

$\Rightarrow \frac{1 - \log x}{x^2} = 0$

or $1 - \log x = 0$ or $\log x = 1 = \log e \Rightarrow x = e$

At $x = e$, the value of

$$y_2 = \frac{2\log e - 3}{(e)^3} = \frac{2.1 - 3}{e^3} = -\frac{1}{e^3} \quad \text{[negative]}$$

$\therefore$ At $x = e$, y is maximum.

Maximum value of function $= \frac{\log e}{e} = \frac{1}{e}$.

12. Let $y = \sin x\,(1 + \cos x) = \sin x + \sin x \cos x$

$\Rightarrow y = \sin x + \frac{1}{2}(2\sin x \cos x) = \sin x + \frac{1}{2}\sin 2x$

$\therefore \frac{dy}{dx} = \cos x + \frac{1}{2}(\cos 2x) \cdot 2 = \cos x + \cos 2x$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}(\cos x + \cos 2x) = -\sin x - 2\sin 2x$

For maxima or minima of y, put $\frac{dy}{dx} = 0$

$\Rightarrow \cos x + \cos 2x = 0$

or $\cos x = -\cos 2x = \cos(\pi - 2x)$

$\therefore x = \pi - 2x$ or $3x = \pi$ or $x = \frac{1}{3}\pi$

When $x = \frac{1}{3}\pi$,

Then the value of $\frac{d^2y}{dx^2} = -\sin\left(\frac{1}{3}\pi\right) - 2\sin\left(\frac{2}{3}\pi\right)$

$$= -\frac{\sqrt{3}}{2} - 2 \times \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2} = \text{negative}$$

Therefore at $x = \frac{\pi}{3}$, the given function is maximum.

Hence proved.

$\therefore$ Maximum value,

$$f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)\left(1 + \cos\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4} = \frac{3^{3/2}}{4}.$$

13. Let $y = \sin x + \cos x$

Then, $\frac{dy}{dx} = \cos x - \sin x$

and $\frac{d^2y}{dx^2} = -\sin x - \cos x$

Now, for maxima or minima, put $\frac{dy}{dx} = 0$

$\Rightarrow \cos x - \sin x = 0 \Rightarrow \sin x = \cos x$

$\Rightarrow \frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = \tan\frac{\pi}{4} \Rightarrow x = \frac{\pi}{4}$

$$\left[\frac{d^2y}{dx^2}\right]_{x = \frac{\pi}{4}} = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0$$

We see that at $x = \frac{\pi}{4}$, the value of $\frac{d^2y}{dx^2}$ is negative.

So, at $x = \frac{\pi}{4}$, y is a maximum function.

So, put $x = \frac{\pi}{4}$ in $y = (\sin x + \cos x)$.

$\therefore$ Maximum value of function

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$ **Hence proved.**

14. Here, $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$

$\Rightarrow f'(x) = 12x^3 + 12x^2 - 24x$

$\Rightarrow f''(x) = 36x^2 + 24x - 24$

For local maxima and local minima, put $f'(x) = 0$

$\Rightarrow 12x^3 + 12x^2 - 24x = 0$

$\Rightarrow 12x(x^2 + x - 2) = 0$

$\Rightarrow 12x(x + 2)(x - 1) = 0$

$\Rightarrow x = 0$ or $x = -2$ or $x = 1$

At $x = 0$, $f''(0) = 0 + 0 - 24 = -24 < 0$

$\therefore f(x)$ is local maximum at $x = 0$

and local maximum value $= f(0) = 0 + 0 - 0 + 12 = 12$

At $x = -2$, $f''(-2) = 36(-2)^2 + 24(-2) - 24$

$= 144 - 48 - 24 = 72 > 0$

$\therefore f(x)$ is local minimum at $x = -2$

and local minimum value $= f(-2)$

$= 3(-2)^4 + 4(-2)^3 - 12(-2)^2 + 12$

$= 48 - 32 - 48 + 12 = -20$

At $x = 1$, $f''(1) = 36 + 24 - 24 = 36 > 0$

$\therefore f(x)$ is local minimum at $x = 1$

and local minimum value $= f(1) = 3 + 4 - 12 + 12 = 7$

15. Given, $y = a\log x + bx^2 + x$

$\therefore \frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

If the value of y is maximum or minimum at $x = -1$ then

$$\left[\frac{dy}{dx}\right]_{x=-1} = 0 \text{ i.e., } \frac{a}{-1} + 2b(-1) + 1 = 0$$

or $-a - 2b + 1 = 0$ ...(1)

If the value of y is maximum or minimum at $x = 2$, then

$$\left[\frac{dy}{dx}\right]_{x=2} = 0 \text{ i.e., } \frac{a}{2} + 2b \times 2 + 1 = 0$$

or $a + 8b + 2 = 0$ ...(2)

Adding eqs. (1) and (2), we get

$$6b + 3 = 0 \Rightarrow b = -\frac{3}{6} = -\frac{1}{2}$$

From eq. (1), $a = 1 - 2b = 1 - 2 \times \left(-\frac{1}{2}\right) = 2$

Hence, $a = 2$, $b = -1/2$.

Long Answer Type Questions

1. Let x be the side and m be the median of an equilateral triangle.

TR!CK

The median of an equilateral triangle is $\frac{\sqrt{3}}{2} \times$ side.

Given, $\frac{dm}{dt} = 2\sqrt{3}$ cm/s

$\because$ Median of an equilateral triangle, $m = \frac{\sqrt{3}}{2}x$

Differentiate w.r.t. t, we get

$$\frac{dm}{dt} = \frac{\sqrt{3}}{2} \cdot \frac{dx}{dt}$$

$$\Rightarrow \quad 2\sqrt{3} = \frac{\sqrt{3}}{2} \cdot \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 4 \text{ cm/s}$$

Hence, the rate at which the side of an equilateral triangle is increasing, is 4 cm/s.

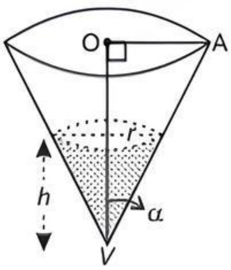
2. Let α be the semi-vertical angle of conical tank.

Then, $\alpha = \tan^{-1}\frac{1}{2}$

$\Rightarrow \tan\alpha = \frac{1}{2}$

Let at any time t, the radius be r, height h and volume V of conical water tank.

Then, $V = \frac{1}{3}\pi r^2 h$



Now, $\tan\alpha = \frac{1}{2} \Rightarrow \frac{r}{h} = \frac{1}{2}$

$\Rightarrow r = \frac{h}{2}$

$\therefore V = \frac{1}{3}\pi r^2 h$

$$\Rightarrow V = \frac{1}{3}\pi \times \left[\frac{h}{2}\right]^2 \times h$$

$$= \frac{1}{3}\pi \times \frac{h^2}{4} \times h = \frac{\pi}{12}h^3$$

TiP

$\frac{dy}{dx}$ is positive if y increases as x increases and is negative if y decreases as x increases.

$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{12} \times 3h^2 \times \frac{dh}{dt} = \frac{\pi h^2}{4} \times \frac{dh}{dt}$$

Given, $\frac{dV}{dt} = 5$ cubic m/min

$$\therefore \frac{\pi h^2}{4} \times \frac{dh}{dt} = 5$$

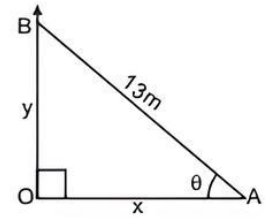
$$\Rightarrow \frac{dh}{dt} = \frac{20}{\pi h^2}$$

$$\Rightarrow \left[\frac{dh}{dt}\right]_{h=4} = \frac{20}{\pi \times 4^2} = \frac{20}{\pi \times 16} = \frac{5}{4\pi}$$

$$= \frac{5}{4} \times \frac{7}{22} = \frac{35}{88} \text{ m/min}$$

Hence, the rate of rising the level of water in tank is $\frac{35}{88}$ m/min when the height of water in tank is 4 m.

3. Let the foot A of ladder AB is at x distance from the wall.



Let the top B of ladder is at height y from the ground.

Let the angle between the foot of ladder and ground be θ.

Then, $x^2 + y^2 = 13^2$ and $\tan\theta = \frac{y}{x}$

$$\Rightarrow 2x \times \frac{dx}{dt} + 2y \times \frac{dy}{dt} = 0$$

and $$(\sec^2\theta) \times \frac{d\theta}{dt} = \frac{x \times \frac{dy}{dt} - y \times \frac{dx}{dt}}{x^2}$$

Given, $\frac{dx}{dt} = 1.5$ m/sec

TiP

Adequate practice is required in problems involving rate of change of bodies.

$$\therefore 2x \times 1.5 + 2y \times \frac{dy}{dt} = 0$$

and $$(\sec^2\theta) \times \frac{d\theta}{dt} = \frac{x \times \frac{dy}{dt} - y \times 1.5}{x^2}$$

$$\Rightarrow \frac{dy}{dt} = \frac{-3x}{2y}$$

and $$(\sec^2\theta) \times \frac{d\theta}{dt} = \frac{x \times \frac{dy}{dt} - \frac{3y}{2}}{x^2}$$

$$\Rightarrow (\sec^2\theta) \times \frac{d\theta}{dt} = \frac{x \times \frac{-3x}{2y} - \frac{3y}{2}}{x^2}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{-3(x^2 + y^2)}{2x^2 y \sec^2\theta}$$

TR!CK

$1 + \tan^2\theta = \sec^2\theta$

$$\Rightarrow \frac{d\theta}{dt} = \frac{-3(x^2 + y^2)}{2x^2 y(1 + \tan^2\theta)}$$

$$= \frac{-3(x^2 + y^2)}{2x^2 y\left[1 + \frac{y^2}{x^2}\right]} \quad \left[\because \tan\theta = \frac{y}{x}\right]$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{-3}{2y} = \frac{-3}{2\sqrt{13^2 - x^2}} \quad [\because x^2 + y^2 = 13^2]$$

$$\therefore \left[\frac{d\theta}{dt}\right]_{x=12} = \frac{-3}{2\sqrt{13^2 - 12^2}} = \frac{-3}{2\sqrt{169 - 144}} = \frac{-3}{2\sqrt{25}}$$

$$= \frac{-3}{2 \times 5} = \frac{-3}{10} \text{ radian/sec}$$

Hence, the angle between ladder and ground is decreasing at the rate of $\frac{3}{10}$ radian/sec when the foot of ladder is 12 m away from the wall.

4.

TiP

Apply product rule to find the derivative of f(x) and then factorise it.

Given, $f(x) = (x-1)^3 (x-2)^2$

$\therefore \quad f'(x) = (x-1)^3 \cdot 2(x-2) + (x-2)^2 \cdot 3(x-1)^2$

$= (x-1)^2 (x-2)(2x-2+3x-6)$

$= (x-1)^2 (x-2)(5x-8)$

The function $f(x)$ will be strictly increasing if $f'(x) > 0$

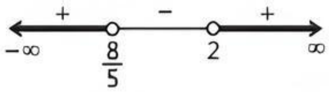
Now, $f'(x) > 0 \Rightarrow (x-1)^2 (x-2)(5x-8) > 0$

$\Rightarrow \quad (x-2)(5x-8) > 0$ and $x \neq 1$

$\Rightarrow \left[x < \frac{8}{5} \text{ or } x > 2\right]$ and $x \neq 1$

$\Rightarrow \quad x \in (-\infty, 1) \cup \left(1, \frac{8}{5}\right)$ or $x \in (2, \infty)$

$\Rightarrow \quad x \in (-\infty, 1) \cup \left(1, \frac{8}{5}\right) \cup (2, \infty)$



Therefore, in the interval $(-\infty, 1) \cup \left(1, \frac{8}{5}\right) \cup (2, \infty)$, $f(x)$ is strictly increasing.

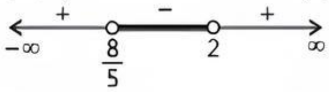
The function $f(x)$ will be strictly decreasing if $f'(x) < 0$.

Now, $f(x) < 0 \Rightarrow (x-1)^2 (x-2)(5x-8) < 0$

$\Rightarrow \quad (x-2)(5x-8) < 0$ and $x \neq 1$

$\Rightarrow \quad \frac{8}{5} < x < 2$ and $x \neq 1$

$\Rightarrow \quad x \in (8/5, 2)$ and $x \neq 1 \Rightarrow x \in (8/5, 2)$



Therefore, in the interval (8/5, 2), $f(x)$ is strictly decreasing.

COMMON ERR(!)R

Some students expand the polynomial before finding the derivative which is not the actual method.

5. Given, $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$

$\therefore \quad f'(x) = \cos x - \sin x = \sqrt{2}\left[\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x\right]$

$= \sqrt{2}\left[\sin \frac{\pi}{4} \cdot \cos x - \cos \frac{\pi}{4} \cdot \sin x\right]$

TR!CK

$\sin A \cdot \cos B - \cos A \cdot \sin B = \sin (A - B)$

$= \sqrt{2} \sin\left[\frac{\pi}{4} - x\right] = -\sqrt{2} \sin\left[x - \frac{\pi}{4}\right]$

The function $f(x)$ will be strictly increasing if $f'(x) > 0$.

Now, $\quad f'(x) > 0 \quad \Rightarrow \quad -\sqrt{2} \sin\left[x - \frac{\pi}{4}\right] > 0$

$\Rightarrow \quad \sin\left[x - \frac{\pi}{4}\right] < 0$

$\Rightarrow \quad \pi < x - \frac{\pi}{4} < 2\pi \quad$ or $\quad x - \frac{\pi}{4} < 0$

$\Rightarrow \quad \pi + \frac{\pi}{4} < x < 2\pi + \frac{\pi}{4} \quad$ or $\quad x - \frac{\pi}{4} < 0$

$\Rightarrow \quad \frac{5\pi}{4} < x < \frac{9\pi}{4} \quad$ or $\quad x < \frac{\pi}{4}$

$\Rightarrow \quad \frac{5\pi}{4} < x \le 2\pi \quad$ or $\quad x < \frac{\pi}{4}$

$\Rightarrow \quad \frac{5\pi}{4} < x \le 2\pi$

or $\quad 0 \le x < \frac{\pi}{4} \quad \because x \in [0, 2\pi]$

$\Rightarrow \quad x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

Therefore, $f(x)$ is strictly increasing in the interval $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$.

The function $f(x)$ will be strictly decreasing if $f(x) < 0$.

Now, $\quad f'(x) < 0$

$\Rightarrow \quad -\sqrt{2} \cdot \sin\left[x - \frac{\pi}{4}\right] < 0$

$\Rightarrow \quad \sin\left[x - \frac{\pi}{4}\right] > 0$

$\Rightarrow \quad 0 < x - \frac{\pi}{4} < \pi$

$\Rightarrow \quad 0 + \frac{\pi}{4} < x < \pi + \frac{\pi}{4}$

$\Rightarrow \quad \frac{\pi}{4} < x < \frac{5\pi}{4}$

$\Rightarrow \quad x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Therefore, $f(x)$ is strictly decreasing in the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

6. Let the sum of two numbers x and y be given *i.e.*, K.

i.e., $\quad x + y = K \quad$ ...(1)

Let the product of these numbers be P.

i.e., $\quad xy = P$

$\Rightarrow \quad P = x(K - x) \quad$ [from eq. (1)]

$\Rightarrow \quad P = Kx - x^2$

Differentiate with respect to x,

$\frac{dP}{dx} = K - 2x$ and $\frac{d^2P}{dx^2} = -2 < 0$

Now, for the maximum or minimum value of P,

put $\quad \frac{dP}{dx} = 0$

$\Rightarrow \quad K - 2x = 0 \quad \Rightarrow \quad x = \frac{K}{2}$

Now, $\left[\frac{d^2P}{dx^2}\right]_{x = \frac{K}{2}} = -2 < 0$

We see that at $x = \frac{K}{2}$, the value of $\frac{d^2P}{dx^2}$ is negative.

$\therefore$ At $x = \frac{K}{2}$, P is maximum.

Put $x = \frac{K}{2}$ in eq. (1), $\frac{K}{2} + y = K \Rightarrow y = \frac{K}{2}$

Therefore, each number *i.e.*, x and y is half of the sum K. **Hence proved.**

7. Let first number be x, then second number will be $5-x$.

Let sum of cubes of these numbers be $S(x)$. Then,

$$S(x) = x^3 + (5-x)^3$$

$\Rightarrow$ $S(x) = x^3 + 5^3 - x^3 - 75x + 15x^2$

$\Rightarrow$ $S(x) = 15x^2 - 75x + 125$

$\Rightarrow$ $S'(x) = 30x - 75$

$\Rightarrow$ $S''(x) = 30$

For least value put, $S'(x) = 0$

$\Rightarrow$ $30x - 75 = 0$

$\Rightarrow$ $x = \frac{75}{30} = \frac{5}{2}$

At $x = \frac{5}{2}$

$$S''(x) = 30 > 0$$

$\therefore$ Local minimum point of S is $x = \frac{5}{2}$.

So, the sum of cubes of these numbers will be least when the numbers are $\frac{5}{2}$ and $\left(5 - \frac{5}{2}\right)$ i.e., $\frac{5}{2}$.

$\therefore$ The sum of the squares of these numbers is

$$\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 = \frac{25}{4} + \frac{25}{4}$$

$$= \frac{25}{2}$$

8. Given, $2x + y = 1$

$\Rightarrow$ $y = 1 - 2x$ ...(1)

Let $u = x^2y = x^2(1-2x)$ [from eq. (1)]

$\Rightarrow$ $u = x^2 - 2x^3$ ...(2)

Then, $\frac{du}{dx} = 2x - 6x^2$ and $\frac{d^2u}{dx^2} = 2 - 12x$

For maximum or minimum of u, put $\frac{du}{dx} = 0$

$\Rightarrow$ $2x - 6x^2 = 0$

$\Rightarrow$ $2x(1-3x) = 0$

$\Rightarrow$ $x = 0, \frac{1}{3}$

$\therefore$ $\left[\frac{d^2u}{dx^2}\right]_{\left(x=\frac{1}{3}\right)} = 2 - 12\left(\frac{1}{3}\right) = 2 - 4 = -2 < 0$

We see that at $x = \frac{1}{3}$, the value of $\frac{d^2u}{dx^2}$ is negative.

$\therefore$ At $x = \frac{1}{3}$, u i.e., given function is maximum.

So, from eq. (2),

Maximum value of function

$$= \left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right)^3 = \frac{1}{9}\left\{1 - 2 \times \frac{1}{3}\right\}$$

$$= \frac{1}{27}$$

9. Let $u = xy^2 = x(10-x)^2$ $[\because x + y = 10 \text{ (given)}]$

$= x(100 + x^2 - 20x) = x^3 - 20x^2 + 100x$ ...(1)

Then, $\frac{du}{dx} = 3x^2 - 40x + 100$

and $\frac{d^2u}{dx^2} = 6x - 40$

Now, for maximum or minimum value of u, put

$$\frac{du}{dx} = 0 \Rightarrow 3x^2 - 40x + 100 = 0$$

TR!CK

$3 \times 100 = 300$

$\because$ $300 = 30 \times 10 = 60 \times 5 = 15 \times 20 = ...$

$\therefore$ *Here we will take 30 and 10 as a factors of 300.*

So, the middle term become $-40 = -30 - 10$

$\Rightarrow$ $3x^2 - 30x - 10x + 100 = 0$

$\Rightarrow$ $3x(x-10) - 10(x-10) = 0$

$\Rightarrow$ $(x-10)(3x-10) = 0$

$\Rightarrow$ $x = 10, \frac{10}{3}$

$\therefore$ $\left[\frac{d^2u}{dx^2}\right]_{x=\frac{10}{3}} = 6 \times \frac{10}{3} - 40$

$$= 20 - 40 = -20 < 0$$

We see that at $x = \frac{10}{3}$, the value of $\frac{d^2u}{dx^2}$ is negative.

$\therefore$ At $x = \frac{10}{3}$, u i.e., given function is maximum.

Therefore from eq. (1),

Maximum value of the function

$$= \left(\frac{10}{3}\right)^3 - 20\left(\frac{10}{3}\right)^2 + 100\left(\frac{10}{3}\right)$$

$$= \frac{(10)^3}{3}\left\{\frac{1}{9} - \frac{2}{3} + 1\right\}$$

$$= \frac{1000}{3} \times \frac{4}{9} = \frac{4000}{27}$$

10. If l is the distance from (2, 1) of any point (x, y) on the curve $y^2 = 4x$.

TiP

The distance between the points (x_1, y_1) and (x_2, y_2) is,

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Then, $l = \sqrt{(x-2)^2 + (y-1)^2}$

If the point (x, y) lies on the curve $y^2 = 4x$, then

$$l = \sqrt{\left(\frac{y^2}{4} - 2\right)^2 + (y-1)^2} \quad \left[\because x = \frac{y^2}{4}\right]$$

$\Rightarrow$ $l^2 = \left(\frac{y^2}{4} - 2\right)^2 + (y-1)^2 = u$ (say)

Now, l is maximum or minimum according as l^2 i.e., u is maximum or minimum.

For a maximum or minimum of u, put $\frac{du}{dy} = 0$.

$\therefore$ $2\left(\frac{y^2}{4} - 2\right)\left(\frac{2y}{4}\right) + 2(y-1) \times 1 = 0$

$\Rightarrow$ $\frac{y^3}{4} - 2y + 2y - 2 = 0$

$\Rightarrow$ $y^3 = 8 = (2)^3$

$\therefore$ $y = 2$

Now, $\frac{d^2u}{dy^2} = \frac{d}{dy}\left(\frac{y^3}{4} - 2\right) = \frac{3y^2}{4}$

At $y = 2$, $\frac{d^2u}{dy^2} = \frac{3}{4}(2)^2 = \frac{3}{4} \times 4 = 3 > 0$

Therefore, u is minimum and hence l is minimum when $y = 2$.

Now, putting the value of y in the equation of the curve, we get

$$x = \frac{(2)^2}{4} = \frac{4}{4} = 1$$

Therefore, the required point is (1, 2).

11. For each value of x, the helicopter's position is at point $(x, x^2 + 7)$. Therefore, the distance between the helicopter and the soldier placed at (3, 7)

$$= \sqrt{(x-3)^2 + (x^2 + 7 - 7)^2}$$

TR!CK

Distance between two points (x_1, y_1) and (x_2, y_2) is,
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$$= \sqrt{(x-3)^2 + x^4}$$

Let $f(x) = (x-3)^2 + x^4$

$\Rightarrow$ $f'(x) = 2(x-3) + 4x^3$

$= 2(x-1)(2x^2 + 2x + 3)$

For maximum or minimum distance,

$f'(x) = 2(x-1)(2x^2 + 2x + 3) = 0$

$\Rightarrow$ $x = 1$

[$\because$ $2x^2 + 2x + 3 = 0$ does not give any real root]

At this value, the value of f is

$$f(1) = (1-3)^2 + (1)^4 = 4 + 1 = 5$$

Thus, the distance between the soldier and the helicopter $= \sqrt{f(1)} = \sqrt{5}$

Note that $\sqrt{5}$ is either a maximum value or minimum value, since

$$\sqrt{f(0)} = \sqrt{(0-3)^2 + (0)^4} = 3 > \sqrt{5}$$

It follows that $\sqrt{5}$ is minimum value of $\sqrt{f(x)}$.

Hence, the minimum distance between the soldier and the helicopter is $\sqrt{5}$.

12.

TR!CKS

- *Volume of cuboid $= l \cdot b \cdot h$. But when its base is square, then volume $= l \cdot l \cdot h = l^2 h$* [$\because$ $l = b$]
- *Surface area of cuboid $= 2(l \cdot b + b \cdot h + h \cdot l)$.*
  But when its base is square, then surface area
  $= 2(l \cdot l + l \cdot h + h \cdot l)$
  $= 2(l^2 + 2lh)$ [$\because$ $l = b$]

Let x be the side of square base of cuboid and other side be y.

Then the volume of a cuboid with square base,

$$V = x \cdot x \cdot y = x^2 y$$

As the volume of the cuboid is given. So, volume is taken constant throughout the question, therefore

$$y = \frac{V}{x^2} \quad ...(1)$$

Let S be the surface area of cuboid, then

$S = 2(x \cdot x + x \cdot y + y \cdot x)$

$= 2(x^2 + 2xy)$

$\Rightarrow$ $S = 2x^2 + 4xy$ ...(2)

$\Rightarrow$ $S = 2x^2 + 4x\left(\frac{V}{x^2}\right)$ [from eq. (1)]

$\Rightarrow$ $S = 2x^2 + \frac{4V}{x}$ ...(3)

Now, differentiate w.r.t. x on both sides;

$$\frac{dS}{dx} = 4x - \frac{4V}{x^2} \quad ...(4)$$

For maximum/minimum value of S, put $\frac{dS}{dx} = 0$

$\Rightarrow$ $4x - \frac{4V}{x^2} \Rightarrow V = x^3$ ...(5)

Putting $V = x^3$ in eq. (1), we get $y = \frac{x^3}{x^2} = x$

Here, $y = x \Rightarrow$ cuboid is a cube.

Now, differentiate eq. (4) w.r.t. x, we have

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3}$$

$\Rightarrow$ $\frac{d^2S}{dx^2} = 4 + \frac{8 \cdot x^3}{x^3} = 8 + 4 = 12 > 0$

[$\because$ when $y = x, V = x^3$]

Hence, surface area is minimum when given cuboid is a cube. **Hence proved.**

13. Let h be the height and r be the radius of base of the cylinder.

If the total surface is S and volume is V of the cylinder, then

$S = 2\pi r^2 + 2\pi rh$ ...(1)

and $V = \pi r^2 h$ ...(2)

From eq. (1), $h = \frac{(S - 2\pi r^2)}{2\pi r}$

Put this value of h in eq. (2),

$$V = \pi r^2 \left\{\frac{S - 2\pi r^2}{2\pi r}\right\} = \frac{1}{2}\{Sr - 2\pi r^3\}$$

Differentiate w.r.t. r, we get

$$\frac{dV}{dr} = \frac{1}{2}(S - 6\pi r^2) \text{ and } \frac{d^2V}{dr^2} = -6\pi r$$

For maximum or minimum of V, put $\frac{dV}{dr} = 0$.

$\Rightarrow$ $S = 6\pi r^2$

or $r = \sqrt{\frac{S}{6\pi}}$

When $r = \sqrt{\frac{S}{6\pi}}$, then $\frac{d^2V}{dr^2} = -\sqrt{6\pi S} = -$ ve.

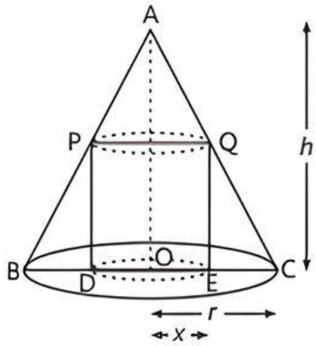
$\therefore$ V is maximum.

From eq. (1),

$6\pi r^2 = 2\pi r^2 + 2\pi rh$ or $2\pi rh = 4\pi r^2$

or $h = 2r$ or height = diameter of base **Hence proved.**

14. Let the radius of base be $OC = r$ and height be $OA = h$ of the cone. Let the radius of circular base of cylinder inscribed in the cone be $OE = x$ and height $= QE$.



$\because \Delta QEC \sim \Delta AOC$

$$\therefore \quad \frac{QE}{OA} = \frac{EC}{OC}$$

$$\Rightarrow \quad \frac{QE}{h} = \frac{r-x}{r}$$

$$\Rightarrow \quad QE = \frac{h(r-x)}{r}$$

Let the curved surface of cylinder be S, then

$$S = S(x) = \frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$

Now, $\quad S'(x) = \frac{2\pi h}{r}(r - 2x)$

and $\quad S''(x) = \frac{2\pi h}{r}(-2) = \frac{-4\pi h}{r}$

For maximum or minimum value of S, put $S'(x) = 0$

$$\therefore \quad \frac{2\pi h}{r}(r-2x) = 0 \Rightarrow x = \frac{r}{2}$$

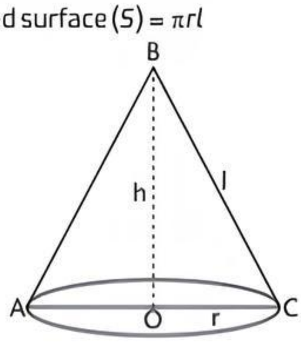
and $\quad (S''(x))_{x=\frac{r}{2}} = \frac{-4\pi h}{r} < 0$

$\therefore x = \frac{r}{2}$ is the maximum point of S.

Hence, the radius of cylinder of maximum curved surface area inscribed in a cone is half of the radius of the cone. **Hence proved.**

15. Let h be the height, l slant height and r radius of the conical tent, then its volume $(V) = \frac{1}{3}\pi r^2 h$ ...(1)

and curved surface $(S) = \pi r l$ ...(2)



In ΔBOC, $\quad l^2 = h^2 + r^2 \quad$ or $\quad l = \sqrt{h^2 + r^2}$

From eq. (2),

$$S = \pi r\sqrt{h^2 + r^2}$$

or $\quad S^2 = \pi^2 r^2 (h^2 + r^2) = U \quad$ (say)

$$U = \pi^2 r^2 \left[\left(\frac{3V}{\pi r^2}\right)^2 + r^2\right] = \frac{9V^2}{r^2} + \pi^2 r^4$$

$$\left[\because \text{ from eq. (1), } h = \frac{3V}{\pi r^2}\right]$$

Now, $\quad \frac{dU}{dr} = 9V^2\left(-\frac{2}{r^3}\right) + 4\pi^2 r^3 = -\frac{18V^2}{r^3} + 4\pi^2 r^3$

and $\quad \frac{d^2U}{dr^2} = -18V^2\left(\frac{-3}{r^4}\right) + 4\pi^2 \cdot (3r^2)$

$$= \frac{54V^2}{r^4} + 12\pi^2 r^2$$

For maximum or minimum of U,

Put $\quad \frac{dU}{dr} = 0 \Rightarrow 4\pi^2 r^3 = \frac{18V^2}{r^3}$ or $18V^2 = 4\pi^2 r^6$

or $\quad V^2 = \frac{2}{9}\pi^2 r^6$ or $V = \frac{1}{3}\sqrt{2} \cdot \pi r^3$

$\because \frac{d^2U}{dr^2}$ is positive at $V = \frac{\sqrt{2}}{3}\pi r^3$,

$\therefore U = S^2$ is minimum

Therefore, S is minimum, when $V = \frac{1}{3}\sqrt{2} \cdot \pi r^3$

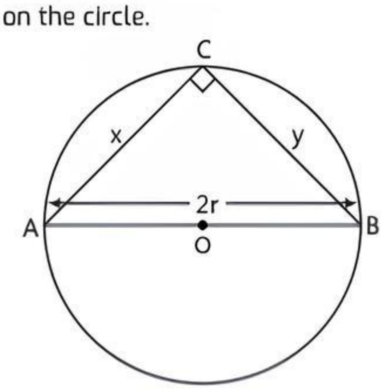
i.e., $\quad \frac{1}{3}\pi r^2 h = \frac{1}{3}\sqrt{2} \cdot \pi r^3$

i.e., $h = \sqrt{2} \cdot r$ or height $= \sqrt{2} \times$ radius of base

Hence, the canvas used in the tent will be minimum when its height is $\sqrt{2}$ times the radius of the base.

Hence proved.

16. Given that, AB is the diameter of a circle and C is any point on the circle.



Let $\quad AB = 2r$

Now, join AC and BC.

So, ΔABC is a right angled triangle.

Let $\quad AC = x$ and $BC = y$

TiP

Angle in a semi-circle is a right angle.

Now, in right ΔACB,

$AB^2 = AC^2 + BC^2 \quad$ (by Pythagoras theorem)

$\Rightarrow \quad (2r)^2 = x^2 + y^2 \Rightarrow y^2 = 4r^2 - x^2 \quad$ ...(1)

TR!CK

In a right-angled triangle, the square of the hypotenuse side is equal to the sum of squares of the other two sides.

and area of $\Delta ACB = A = \frac{1}{2} AC \times BC$

$$\left[\because \text{area of triangle} = \frac{1}{2} \times \text{base} \times \text{height}\right]$$

$\Rightarrow \quad A = \frac{1}{2} xy$

On squaring both sides, we get

$$A^2 = \frac{1}{4} x^2y^2 = \frac{1}{4} x^2 (4r^2 - x^2) \quad \text{[from eq. (1)]}$$

Let $\quad S = A^2 = \frac{1}{4}(4x^2r^2 - x^4) \quad \text{...(2)}$

Here, A^2 is maximum or minimum according to S is maximum or minimum.

Now, differentiate eq. (2) w.r.t. x on both sides,

$$\frac{dS}{dx} = \frac{1}{4}\{8xr^2 - 4x^3\}$$

For maximum or minimum of S, put $\frac{dS}{dx} = 0$

$\Rightarrow \quad \frac{1}{4}(8xr^2 - 4x^3) = 0$

$\Rightarrow \quad 4x(2r^2 - x^2) = 0$

$\therefore \quad x^2 = 2r^2$

$\Rightarrow \quad x = \sqrt{2}r \quad [\because x \neq 0]$

Put the value of x in eq. (1), we get

$$y^2 = 4r^2 - (2r^2) = 2r^2$$

$\therefore \quad y = \sqrt{2}\, r$

i.e., $\quad x = y = \sqrt{2}\, r$

Now, $\quad \frac{d^2S}{dx^2} = \frac{1}{4}(8r^2 - 12x^2)$

$$\therefore \left[\frac{d^2S}{dx^2}\right]_{x = \sqrt{2}r} = \frac{1}{4}(8r^2 - 12 \times 2r^2)$$

$$= \frac{1}{4}(8r^2 - 24r^2)$$

$$= \frac{1}{4} \times -16r^2 = -4r^2 < 0$$

So, area is maximum, when $x = y = \sqrt{2}r$ *i.e.*, the area of triangle ABC is maximum, when it is an isosceles triangle. **Hence proved.**

17. Let the length, breadth and height of the metal box be x cm, x cm and y cm respectively.

It is given that the box can contain 1024 cm^3.

$\therefore \quad 1024 = x^2y$

$\Rightarrow \quad y = \frac{1024}{x^2} \quad \text{...(1)}$

TR!CK

Volume of cuboid $= l \cdot b \cdot h$

When the base of the cuboid is square, then its volume $= l \cdot l \cdot h \quad [\because l = b]$

Let C be the cost in rupees of the material used to construct.

Then, $C = 5 \times (2 \times x \times x) + 2.5 \times (2 \times x \times y) + 2.5 \times (2 \times x \times y)$

$\Rightarrow \quad C = 10x^2 + 5xy + 5xy$

$\Rightarrow \quad C = 10x^2 + 10xy$

$\Rightarrow \quad C = 10x^2 + 10x\left(\frac{1024}{x^2}\right) \quad \text{[from eq. (1)]}$

$\Rightarrow \quad C = 10x^2 + \frac{10240}{x} \quad \text{...(2)}$

TR!CK

Surface area of cuboid $= 2(l \cdot b + b \cdot h + h \cdot l)$

But when the base of the cuboid is square, then its surface area $= 2(l \cdot l + l \cdot h + h \cdot l) \quad [\because l = b]$

Now, differentiate w.r.t. 'x' on both sides, we have

$$\frac{dC}{dx} = 20x - \frac{10240}{x^2}$$

For maxima or minima of C, put $\frac{dC}{dx} = 0$

$\Rightarrow \quad 20x - \frac{10240}{x^2} = 0$

$\Rightarrow \quad x^3 = 512 = (8)^3 \Rightarrow x = 8$

Again $\quad \frac{d^2C}{dx^2} = 20 + \frac{20480}{x^3}$

$$\Rightarrow \left[\frac{d^2C}{dx^2}\right]_{(\text{at } x = 8)} = 20 + \frac{20480}{(8)^3} = 20 + \frac{20480}{512}$$

$$= 20 + 40 = 60 > 0$$

Thus, the cost of the box is least when $x = 8$ cm.

Put $x = 8$ in eq. (1), we get

$$y = \frac{1024}{(8)^2} = \frac{1024}{64} = 16$$

So, dimensions of the box are x, x, y or 8 cm, 8 cm, 16 cm.

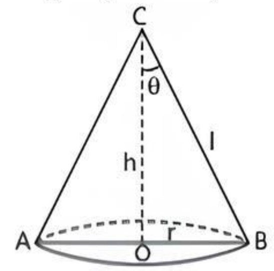
Put $x = 8$ in eq. (2), we get

$$C = 10(8)^2 + \frac{10240}{8} = 10 \times 64 + 1280$$

$$= 640 + 1280 = 1920.$$

Hence, the least cost of the box is ₹ 1920.

18. Let r be the radius and h be the height of the cone of given slant height l (constant).



Then $\quad l^2 = h^2 + r^2$

$\Rightarrow \quad r^2 = l^2 - h^2 \quad \text{...(1)}$

Let volume of cone be V.

Then, $\quad V = \frac{1}{3}\pi r^2 h$

or $\quad V = \frac{1}{3}\pi(l^2 - h^2)h \quad \text{[from eq. (1)]}$

or $\quad V = \frac{1}{3}\pi(l^2h - h^3)$

Differentiate both sides with respect to h,

$$\frac{dV}{dh} = \frac{1}{3}\pi(l^2 - 3h^2)$$

Again differentiate with respect to h,

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(0 - 6h) = -2\pi h$$

Now, for maximum or minimum of V; put $\frac{dV}{dh} = 0$

$\Rightarrow \quad \frac{1}{3}\pi(l^2 - 3h^2) = 0$

$\Rightarrow \quad l^2 - 3h^2 = 0$

$\Rightarrow \quad l^2 = 3h^2$

$\Rightarrow \quad h^2 = \frac{l^2}{3}$

$\Rightarrow \quad h = \pm\frac{l}{\sqrt{3}}$

[but $h = \frac{-l}{\sqrt{3}}$ is not possible as height cannot be negative]

$\therefore$ At $h = \frac{l}{\sqrt{3}}$, $\frac{d^2V}{dh^2} = -2\pi \times \frac{l}{\sqrt{3}} = -\frac{2\pi l}{\sqrt{3}}$

We see that at $h = \frac{l}{\sqrt{3}}$, the value of $\frac{d^2V}{dh^2}$ is negative, so V is maximum at $h = \frac{l}{\sqrt{3}}$

Put $h = \frac{l}{\sqrt{3}}$ in eq. (1), $r^2 = l^2 - \frac{l^2}{3} = \frac{2l^2}{3} \Rightarrow r = \frac{l\sqrt{2}}{\sqrt{3}}$

If the semi-vertical angle of cone is θ, then in right ΔBOC,

$$\tan\theta = \frac{r}{h} = \frac{\frac{l\sqrt{2}}{\sqrt{3}}}{\frac{l}{\sqrt{3}}} = \frac{l\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{l} = \sqrt{2}$$

or $\quad \theta = \tan^{-1}\sqrt{2}$

Hence, the semi-vertical angle of cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.

Hence proved.

COMMON ERROR

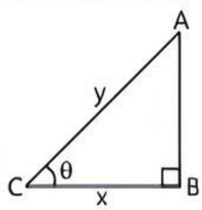
Most of the students attempt this question incorrectly. Some students could not express volume of the cone as a function in mathematical form.

19. Let ABC be the right angled triangle with $BC = x$, $AC = y$ such that

$$x + y = k \quad \text{...(1)}$$

where, k is any constant.

Let θ be the angle between the base and the hypotenuse.



TRICK

Area of triangle = $\frac{1}{2}$ × base × height

Let A be the area of the triangle.

$\therefore \quad A = \frac{1}{2} \times BC \times AB \quad \text{...(2)}$

TRICK

In a right-angled triangle, the square of the hypotenuse side is equal to the sum of squares of the other two sides.

Now, in right ΔABC,

$AC^2 = BC^2 + AB^2$ [by Pythagoras theorem]

$\Rightarrow \quad y^2 = x^2 + AB^2 \quad \Rightarrow AB = \sqrt{y^2 - x^2}$

From eq. (2), we get

$$A = \frac{1}{2} \times x \times \sqrt{y^2 - x^2}$$

$\Rightarrow \quad A^2 = \frac{x^2}{4}(y^2 - x^2)$

$\Rightarrow \quad A^2 = \frac{1}{4}x^2\{(k - x)^2 - x^2\}$ [from eq. (1)]

$\Rightarrow \quad A^2 = \frac{x^2}{4}(k^2 + x^2 - 2kx - x^2)$

$\therefore \quad A^2 = \frac{x^2}{4}(k^2 - 2kx)$

Let $\quad S = A^2 = \frac{x^2}{4}(k^2 - 2kx) \quad \text{...(3)}$

Here A^2 is maximum or minimum according to S is maximum or minimum.

$\because \quad S = \frac{1}{4}(k^2x^2 - 2kx^3)$

Differentiate w.r.t. x, we get

$$\frac{dS}{dx} = \frac{1}{4}(2k^2x - 6kx^2) \quad \text{...(4)}$$

Now, for maximum or minimum of S; put $\frac{dS}{dx} = 0$

$\therefore \quad \frac{1}{4}(2k^2x - 6kx^2) = 0$

$\Rightarrow \quad 2kx(k - 3x) = 0$

$\Rightarrow \quad k - 3x = 0 \quad [\because x \neq 0, k \neq 0]$

$\Rightarrow \quad x = \frac{k}{3}$

Again differentiate eq. (4) w.r.t. x, we get

$$\frac{d^2S}{dx^2} = \frac{1}{4}(2k^2 - 12kx)$$

$\therefore \quad \left[\frac{d^2S}{dx^2}\right]_{x=\frac{k}{3}} = \frac{1}{4}\left\{2k^2 - 12k \times \frac{k}{3}\right\} = \frac{1}{4}(2k^2 - 4k^2)$

$$= -\frac{k^2}{2} < 0$$

So, S as well as A is maximum when $x = \frac{k}{3}$

Put the value of x in eq. (1), we get

$$\frac{k}{3} + y = k \Rightarrow y = \frac{2k}{3}$$

Now, in right ΔABC,

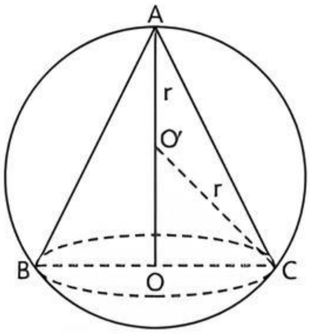
$$\cos\theta = \frac{BC}{AC} = \frac{x}{y}$$

$\Rightarrow \quad \cos\theta = \frac{k/3}{2k/3} = \frac{1}{2} = \cos 60°$

$\Rightarrow \quad \theta = 60° \quad$ or $\quad \theta = \frac{\pi}{3}$ **Hence proved.**

20. Let O' be the centre of the sphere, then

$$O'A = O'C = r$$



Let h be the height of cone ABC.

$\therefore$ $OA = h$, then $OO' = h - r$

TR!CK

In a right-angled triangle, the square of the hypotenuse side is equal to the sum of squares of the other two sides.

In right angled $\Delta O'OC$,

$$OC = \sqrt{O'C^2 - OO'^2} \quad \text{[by Pythagoras theorem]}$$
$$= \sqrt{r^2 - (h-r)^2} = \sqrt{2rh - h^2}$$

If V is the volume of cone, then

$$V = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi (OC)^2 \cdot OA$$
$$= \frac{1}{3}\pi(2rh - h^2)\cdot h = \frac{1}{3}\pi(2rh^2 - h^3)$$

Differentiate w.r.t. h,

$$\frac{dV}{dh} = \frac{1}{3}\pi(4rh - 3h^2)$$

and $$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4r - 6h)$$

For maximum or minimum of V,

Put $\frac{dV}{dh} = 0 \Rightarrow \frac{1}{3}\pi(4rh - 3h^2) = 0 \Rightarrow h = \frac{4}{3}r$

When $h = \frac{4}{3}r$, then $\frac{d^2V}{dh^2} = -\frac{4}{3}\pi r$ (negative)

Then volume V of the cone will be maximum when $h = \frac{4}{3}r$.

i.e., $\frac{h}{r} = \frac{4}{3}$

Thus, the ratio of the height of cone and radius of sphere is $h : r = 4 : 3$.

Hence, it can be seen that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. **Hence proved.**

And maximum volume $(V) = \frac{1}{3}\pi(2rh - h^2)\cdot h$

$$= \frac{1}{3}\pi\left(2r\cdot\frac{4r}{3} - \frac{16r^2}{9}\right)\cdot\frac{4r}{3}$$
$$= \frac{1}{3}\pi\left(\frac{8r^2}{3} - \frac{16r^2}{9}\right)\cdot\frac{4r}{3}$$
$$= \frac{1}{3}\pi \times \frac{8r^2}{9} \times \frac{4r}{3} = \frac{32\pi}{81}r^3$$

21. Suppose the wire is cut into two pieces of length x and y meters respectively.

So, $$x + y = 34 \quad ...(1)$$

Perimeter of square $= 4 \times (\text{Side})$

$= \text{Length of the first piece}$

$\Rightarrow$ $4 \times (\text{Side}) = x$

TiP

Obtaining the result in the required form needs simplification, cross-multiplication, substitution etc.

$\Rightarrow$ $\text{Side} = \frac{x}{4}$

So, Area of square $= (\text{Side})^2 = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$

and perimeter of rectangle $= 2 \times (\text{Length} + \text{Breadth})$

$= \text{Length of the second piece}$

$\Rightarrow$ $2(\text{Length} + \text{Breadth}) = y$

$\Rightarrow 2(2 \times \text{Breadth} + \text{Breadth}) = y$

$[\because \text{length} = 2 \times \text{breadth}]$

$\Rightarrow$ $3 \times \text{Breadth} = \frac{y}{2}$

$\therefore$ $\text{Breadth} = \frac{y}{6}$

So, Area of rectangle $= \text{Length} \times \text{Breadth}$

$= 2 \times \text{Breadth} \times \text{Breadth}$

$= 2 \times \frac{y}{6} \times \frac{y}{6} = \frac{y^2}{18}$

Let combined area (A) = Area of square + Area of rectangle

$\Rightarrow$ $A = \frac{x^2}{16} + \frac{y^2}{18} \Rightarrow A = \frac{x^2}{16} + \frac{(34-x)^2}{18}$

[from eq. (1)]

Differentiate w.r.t. 'x', we get

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{2(34-x)}{18} \times (-1) = \frac{x}{8} - \frac{(34-x)}{9}$$
$$= \frac{9x - 272 + 8x}{72} = \frac{17x - 272}{72}$$

For maxima or minima of A, put

$$\frac{dA}{dx} = 0 \Rightarrow \frac{17x - 272}{72} = 0$$

$\Rightarrow$ $x = \frac{272}{17} = 16$

From eq. (1), we get

$$16 + y = 34$$

$\Rightarrow$ $y = 34 - 16 = 18$

Now, $\frac{d^2A}{dx^2} = \frac{1}{72} \times 17 > 0$

Hence, A is minimum when $x = 16$ and $y = 18$.

So, the wire should be cut into two pieces of length 16 m and 18 m.

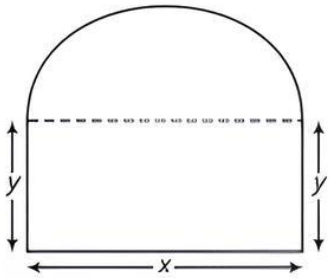
COMMON ERR!R

The required sum of areas of the two parts has to be expressed correctly in terms of a single variable. Sometimes students fail to do this.

22. Let length of rectangle be x and breadth of rectangle be y.

From figure:

Diameter of semi-circle = Length of rectangle = x



$\therefore$ Radius of rectangle $(r) = \dfrac{x}{2}$

TiP

Give ample practice to the problems based on maxima and minima.

Given, perimeter of window = 10 m

$\therefore$ Length of rectangle + 2 × Breadth of rectangle + Circumference of semi-circle = 10

$$\Rightarrow \quad x + 2y + \pi r = 10$$

$$\Rightarrow \quad x + 2y + \pi \frac{x}{2} = 10$$

TR!CK

Circumference of semi-circle = π (radius)

$$\Rightarrow \quad 2y = 10 - x - \frac{\pi x}{2}$$

$$\Rightarrow \quad y = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right) \qquad \text{...(1)}$$

We need to maximize area of window.

Area of window = Area of rectangle + Area of semi-circle

TR!CKS

- *Area of rectangle = length × breadth*
- *Area of semi-circle = $\frac{1}{2}$ × (radius)^2*

$$\Rightarrow \quad A = xy + \frac{1}{2}\pi r^2$$

$$\Rightarrow \quad A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$\Rightarrow \quad A = x\left\{5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right\} + \frac{1}{2}\pi\left(\frac{x^2}{4}\right)$$

[from eq. (1)]

$$\Rightarrow \quad A = 5x - \frac{x^2}{2} - \frac{\pi x^2}{4} + \frac{\pi x^2}{8}$$

$$\Rightarrow \quad A = 5x - \frac{x^2}{2} - \frac{\pi x^2}{8}$$

Differentiate w.r.t. x on both sides,

$$\frac{dA}{dx} = 5 - \frac{2x}{2} - \frac{2\pi x}{8} = 5 - x - \frac{\pi x}{4}$$

For maximum or minimum of A, put $\dfrac{dA}{dx} = 0$

$$\Rightarrow \quad 5 - x - \frac{\pi x}{4} = 0$$

$$\Rightarrow \quad x\left(1 + \frac{\pi}{4}\right) = 5$$

$$\Rightarrow \quad x = \frac{20}{\pi + 4}$$

Again differentiate w.r.t. x on both sides,

$$\frac{d^2A}{dx^2} = -1 - \frac{\pi}{4} < 0$$

$\therefore$ At $x = \dfrac{20}{\pi + 4}$, $\dfrac{d^2A}{dx^2} < 0$

So, A is maximum when $x = \dfrac{20}{\pi + 4}$.

From eq. (1), we get

$$y = 5 - \frac{20}{\pi + 4} \times \frac{\pi + 2}{4} = 5 - \frac{5\pi + 10}{\pi + 4}$$

$$= \frac{5\pi + 20 - 5\pi - 10}{\pi + 4} = \frac{10}{\pi + 4}$$

Hence, the dimensions of the window to admit maximum light through the whole opening are $\dfrac{20}{\pi + 4}$ m and $\dfrac{10}{\pi + 4}$ m.

23. Let l, b and h be the length, breadth and height of the tank respectively.

TR!CK

Volume of cuboid = $l \cdot b \cdot h$

Given, height (depth) of the tank $(h) = 2$ m

and volume of the tank = 8 m^3

$$\therefore \quad l \times b \times h = 8$$

$$\Rightarrow \quad l \times b \times 2 = 8$$

$$\Rightarrow \quad lb = 4$$

$$\Rightarrow \quad b = \frac{4}{l} \qquad \text{...(1)}$$

Also, area of the base = $lb = l \times \dfrac{4}{l} = 4$.

and area of the four sides = $2l + 2b + 2l + 2b$

$= 4l + 4b = 4(l + b)$

It is given that, the cost of construction on base is ₹ 70 per sq. metre and for side is ₹ 45 per sq. metre.

So, cost of construction, C = ₹ $[70 \times lb + 45 \times 4(l + b)]$

= ₹ $[70lb + 180(l + b)]$ ...(2)

On putting the value of b in eq. (2) from eq. (1), we get

$$C = 70 \times 4 + 180\left(l + \frac{4}{l}\right)$$

$$= 280 + 180\left(l + \frac{4}{l}\right) \qquad \text{...(3)}$$

On differentiating w.r.t. x, we get

$$\frac{dc}{dl} = 180\left(1 - \frac{4}{l^2}\right) = 180\left(\frac{l^2 - 4}{l^2}\right)$$

For least expensive, put $\frac{dc}{dl}=0$

$$\Rightarrow \quad 180\left(\frac{l^2-4}{l^2}\right)=0 \Rightarrow l^2=4 \Rightarrow l=\pm 2$$

$\frac{dc}{dl}$ changes sign from negative to positive at $l=2$.

$\therefore$ C is minimum at $l=2$.

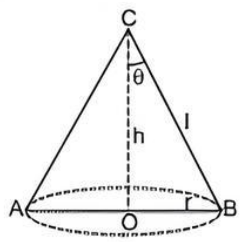
[$\because$ length of the tank cannot be negative, so, $l=-2$ is not consider]

$$l=2 \text{ and } b=\frac{4}{l}=\frac{4}{2}=2.$$

Thus, tank is a cube of side 2 m.

$$\text{Least cost of construction} = ₹\left[280+180\left(2+\frac{4}{2}\right)\right]$$
$$=₹(280+720)$$
$$=₹1000 \quad \text{[from eq. (3)]}$$

24. Let S be the total surface area of the given cone with radius r, height h and slant height l.



Then, $S=\pi r^2+\pi rl$

or $\quad \pi rl = S-\pi r^2$

or $\quad l=\frac{S-\pi r^2}{\pi r}$ ...(1)

Let V be the volume of cone.

Then, $\quad V=\frac{1}{3}\pi r^2 h$

or $\quad V^2=\frac{1}{9}\pi^2 r^4 h^2=\frac{1}{9}\pi^2 r^4(l^2-r^2)$

[$\because$ in right ΔBOC, $l^2=h^2+r^2$]

TiP

Explain exhaustively the concept of maxima and minima and its application. Give practice in problems based on maxima and minima.

or $\quad V^2=\frac{1}{9}\pi^2 r^4\left\{\left\{\frac{S-\pi r^2}{\pi r}\right\}^2-r^2\right\}$ [from eq. (1)]

or $\quad V^2=\frac{1}{9}\pi^2 r^4 \times \frac{(S^2+\pi^2 r^4-2\pi S r^2)-\pi^2 r^4}{\pi^2 r^2}$

or $\quad V^2=\frac{1}{9}r^2(S^2-2\pi S r^2)=\frac{S}{9}(Sr^2-2\pi r^4)$

Let $\quad u=V^2=\frac{S}{9}(Sr^2-2\pi r^4)$

Then $\quad \frac{du}{dr}=\frac{S}{9}(2Sr-8\pi r^3)$

and $\quad \frac{d^2u}{dr^2}=\frac{S}{9}(2S-24\pi r^2)$

For maximum or minimum value of u, put $\frac{du}{dr}=0$.

$$\Rightarrow \quad \frac{S}{9}(2Sr-8\pi r^3)=0$$
$$\Rightarrow \quad 2Sr-8\pi r^3=0$$
$$\Rightarrow \quad 2r(S-4\pi r^2)=0$$
$$\Rightarrow \quad S=4\pi r^2 \quad [\because r\neq 0]$$
$$\Rightarrow \quad 4\pi r^2=S$$
$$\Rightarrow \quad r^2=\frac{S}{4\pi}$$

$\therefore$ At $r^2=\frac{S}{4\pi}$,

$$\frac{d^2u}{dr^2}=\frac{S}{9}\left[2S-24\pi\times\frac{S}{4\pi}\right]$$
$$=\frac{S}{9}(2S-6S)=\frac{S}{9}(-4S)=\frac{-4S^2}{9}$$

We see that at $r^2=\frac{S}{4\pi}$, the value of $\frac{d^2u}{dr^2}$ is negative.

$\therefore$ At $r^2=\frac{S}{4\pi}\cdot u$ i.e., V^2 i.e., V is maximum.

Put $r^2=\frac{S}{4\pi}$ i.e., $S=4\pi r^2$ in eq. (1),

$$l=\frac{4\pi r^2-\pi r^2}{\pi r}=\frac{3\pi r^2}{\pi r}=3r$$

If the semi-vertical angle of cone is θ then in right ΔBOC:

$$\sin\theta=\frac{OB}{BC}=\frac{r}{l}=\frac{r}{3r}=\frac{1}{3}$$

or $\quad \theta=\sin^{-1}\frac{1}{3}$

Hence, the volume of cone with given total surface area will be maximum, if its semi-vertical angle is $\sin^{-1}\frac{1}{3}$. **Hence proved.**

Chapter Test

Multiple Choice Questions

Q 1. Moving along the X-axis, there are two points with $x=10+6t$, $x=3+t^2$. The speed with which they are reaching from each other at the time of encounter is: (x is in cm and t is in seconds)

a. 16 cm/s b. 20 cm/s

c. 8 cm/s d. 12 cm/s

Q 2. The interval in which the function $y=x^3+5x^2-1$ is decreasing, is:

a. $\left(0, \frac{10}{3}\right)$ b. (0, 10)

c. $\left(\frac{-10}{3}, 0\right)$ d. None of these

Assertion and Reason Type Questions

Directions (Q. Nos. 3-4): *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation for Assertion (A)
b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)
c. Assertion (A) is true but Reason (R) is false
d. Assertion (A) is false but Reason (R) is true

Q 3. Assertion (A): $y = \frac{e^x + e^{-x}}{2}$ is an increasing function on $[0, \infty)$.

Reason (R): $y = \frac{e^x - e^{-x}}{2}$ is an increasing function on $(-\infty, \infty)$.

Q 4. Assertion (A): If manufacturer can sell x items at a price of ₹$\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is ₹ $\left(\frac{x}{5} + 500\right)$. Then, the number of items he should sell to earn maximum profit is 240 items.

Reason (R): The profit for selling x items is given by $\frac{24}{5}x - \frac{x^2}{100} - 300$.

Case Study Based Questions

Q 5. Case Study 1

Shreya got a rectangular parallelopiped shaped box and spherical ball inside it as return gift. Sides of the box are $x, 2x$ and $x/3$, while radius of the ball is r.

Based on the above information, solve the following questions:

(i) If S represents the sum of volume of parallelopiped and sphere, then find the expression of S.

(ii) If sum of the surface area of box and ball are given to be constant k^2, then find the value of x.

(iii) Find the radius of the ball in terms of x, when S is minimum.

Or

Find the minimum value of S.

Q 6. Case Study 2

During rainy season, a lot of water is wasted. To prevent it, village Panchayat decides to dug out a square tank of capacity 1000 cubic metres. The cost of land is ₹100 per m^2. The cost of digging increases with the depth and for the whole tank, it is ₹$(3200 \times h^2)$, where h metres is the depth of the tank.

Based on the above information, solve the following questions:

(i) Find the total cost of the land in terms of h.

(ii) Village panchayat wants minimum cost for making tank, then find the value of h.

(iii) For minimum cost, find the value of x.

Or

Find the cost of digging for the whole tank.

Very Short Answer Type Questions

Q 7. The function f defined by $f(x) = 4x^4 - 2x + 1$, for what value of x function $f(x)$ should be increasing?

Q 8. Find the maximum value of the function $f(x) = 3x^2 + 6x + 8, x \in R$.

Short Answer Type-I Questions

Q 9. Find the value of b for which the function $f(x) = \sin x - bx + c$, is decreasing for $x \in R$.

Q 10. The radius of a cylinder is increasing at the rate of 3 m/s and its height is decreasing at the rate of 4 m/s. Find the rate of change of volume when radius is 4 m and height is 6 m.

Short Answer Type-II Questions

Q 11. Prove that the function given by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in (–1, 1).

Q 12. Find two numbers whose sum is 6 and the sum of whose cubes is minimum.

Long Answer Type Questions

Q 13. Prove that among all the rectangles inscribed in a circle, the area of square is maximum.

Q 14. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by near by settled lower income families, for whom water will be provided?